AN EQUIVALENCE OF APPROXIMATE REASONING UNDER DEFUZZIFICATION

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Abstract. A specific type of equivalence of inference results under defuzzification between logical implication interpretation and conjunctive interpretation of the fuzzy if-then rules has been considered in this paper.

1. Introductory remarks

A study of inference processes when premises in if-then rules are fuzzy is still a subject of many papers in specialized literature c.f. [1-9]. In such processes, a sound and proper choice of logical operators plays an essential role. The theoretical (mathematical) and the practical (computational) behavior of logical operators in inference processes has to be known before such a choice is made.

Some selected logical operators and fuzzy implications were investigated here with respect to their behavior in the inference processes. A specific type of equivalence of inference results obtained using on one hand a conjunction interpretation of fuzzy if-then rules and on the other hand the interpretation of such rules in the spirit of classical logical implication may be shown. Such equivalence is important regarding the inference algorithms construction. The inference algorithms based on logical implication interpretation of the if-then rules may be replaced by the simpler, faster and more exact algorithms used for the conjunctive interpretation of such rules.

2. Approximate reasoning using generalized modus ponens and fuzzy implications

Fuzzy implications are mostly used as a way of interpretation of the if-then rules with fuzzy antecedent and/or fuzzy consequent. Such rules constitute a convenient form of expressing pieces of knowledge and a set of if-then rules forms a fuzzy rule base. Let us consider the canonical form of fuzzy if-then rule R^k, which includes other types of fuzzy rules and fuzzy propositions as special cases, in the (MISO) form

$$R^k:\ if\, X_1\ is\ A_1^k\ and \ldots and\ X_n\ is\ A_n^k\ then\ Y\ is\ B^k \tag{1}$$

where X_i and Y stand for linguistic variables of the antecedent and consequent and A_i^k, B^k are fuzzy sets in universes of discourse $\mathbb{X}_i \subset \mathbb{R}$, $\mathbb{Y} \subset \mathbb{R}$ respectively.
Such a linguistic form of fuzzy if-then rule can be also expressed as a fuzzy relation:

$$R^k = (A_1^k \times \ldots \times A_n^k \Rightarrow B^k) = (\underline{A}^k \Rightarrow B^k) \tag{2}$$

where $\underline{A}^k = A_1^k \times \ldots \times A_n^k$ is a fuzzy relation in $\mathbb{X} = \mathbb{X}_1 \times \ldots \times \mathbb{X}_n$ defined by

$$(A_1^k \times \ldots \times A_n^k)(x_1, \ldots, x_n) = A_1^k(x_1) \star_T \ldots \star_T A_n^k(x_n) = \underline{A}^k(\underline{x}) \tag{3}$$

where $\star_T$ - denotes respective t-norm T.

Fuzzy if-then rules may be interpreted in two ways: as a conjunction of the antecedent and the consequent (Mamdani combination) or in the spirit of the classical logical implication i.e. as a fuzzy implication [3,4,9].

Approximate reasoning is usually executed in a fuzzy inference system which performs a mapping from an input fuzzy set $\underline{A}'$ in $\mathbb{X}$ to a fuzzy set B' in $\mathbb{Y}$ via a fuzzy rule base. Two methods of approximate reasoning are mostly used: composition based inference (first aggregate then inference - FATI) and individual-rule based inference (first inference then aggregate - FITA).

In composition based inference, a finite number of rules k = 1,..., K is aggregated via intersection or average operations i.e.

$$R = \bigcap_{k=1}^{K}{}_{T,\Sigma} R^k \tag{4}$$

where $\cap_{T,\Sigma}$ denotes the symbol of connective operation using t-norm (T) or averages (e.g. normalized arithmetic sum (Σ/K)) for aggregation of respective membership functions

$$R(\underline{x}, y) = R^1(\underline{x}, y) \begin{bmatrix} \star_T \\ + \end{bmatrix} \dots \begin{bmatrix} \star_T \\ + \end{bmatrix} R^K(\underline{x}, y) \tag{5}$$

Taking into account an arbitrary input fuzzy set $\underline{A}'$ in $\mathbb{X}$ and using the generalized modus ponens we obtain the output of fuzzy inference (FATI):

$$B' = \underline{A}' \circ R = \underline{A}' \circ \bigcap_{k=1}^{K}{}_{T,\Sigma} R^k = \underline{A}' \circ \bigcap_{k=1}^{K}{}_{T,\Sigma} (\underline{A}^k \Rightarrow B^k) \tag{6}$$

or in terms of membership functions:

$$B'(y) = \sup_{\underline{x} \in X} \star_{T'} [\underline{A}'(\underline{x}), R(\underline{x}, y)] = \sup_{\underline{x} \in X} \star_{T'} [\underline{A}'(\underline{x}), \begin{bmatrix} \bigwedge_{k=1}^{K}{}_T \\ \sum_{k=1}^{K} \end{bmatrix} R^k(\underline{x}, y)] \tag{7}$$

where $\wedge_T$, $\star_T$ denote t-norms (T, T') for aggregation operation and composition respectively.

In individual - rule based inference (FITA) each rule in the fuzzy rule base determines an output fuzzy set and after that an aggregation via intersection or average operation is performed. So the output fuzzy set is expressed by means of the formulas:

$$B'' = \bigcap_{k=1}^{K}{}_{T,\Sigma} \underline{A}' \circ (\underline{A}^k \Rightarrow B^k) \tag{8}$$

or:

$$B''(y) = \begin{bmatrix} \bigwedge_{k=1}^{K}{}_T \\ \sum_{k=1}^{K} \end{bmatrix} \sup_{\underline{x} \in X} \star_{T'} [\underline{A}'(\underline{x}), R^k(\underline{x}, y)] \tag{9}$$

It can be proved that B' is more specified then B" i.e.

$$B' \subseteq B'' \quad or \quad B'(y) \leq B''(y) \tag{10}$$

It means that the consequent B' is equal to or contained in the intersection of fuzzy inference results - B". Sometimes for simplicity of calculation the consequent B' is replaced by B", under the assumption that the differences are not so big.

If the input fuzzy sets $A'_1,\dots,A'_n$ or ($\underline{A}'$) are singletons in $x_{10},\dots,x_{n0}$ or ($\underline{x}_0$), the consequence B' is equal to B" (B'(y) = B"(y)).

Fuzzy inference engine generate inference results based on fuzzy if-then rules. These results are mainly fuzzy sets. In many applications crisp results are required instead of fuzzy ones. The transformation of fuzzy results into crisp is performed by a defuzzification method generally expressed as

$$\delta : \mathcal{F}(Y) \to Y \tag{11}$$

or equivalently for membership functions

$$\delta_{mf} : \{ B'(y) \mid B' \in \mathcal{F}(Y), y \in Y \} \to Y \tag{12}$$

where symbols δ, δ_{mf} stand for defuzzification mapping.

It means that defuzzification is defined as a mapping from fuzzy set B' which is the output of the fuzzy inference to crisp point $y^* \in \mathbb{Y}$. This also implies that a defuzzification method should be chosen. Although many various defuzzification methods (defuzzifiers) may be proposed, we will recall the most important ones from the class of standard defuzzifiers (SD).

1. The center of gravity (COG) defuzzifier specifying the y* as the center of the area covered by the membership function of B' i.e.

$$y^* = \frac{\int_Y y B'(y)\, dy}{\int_Y B'(y)\, dy} \tag{13}$$

2. Sometimes it is necessary to eliminate the $y \in \mathbb{Y}$, whose membership values in B' are too small or equal in all $\mathbb{Y}$ (non-informative part of membership function). In this case we use the indexed center of gravity (ICOG) defuzzifier which results in:

$$y^* = \frac{\int_{Y_\alpha} y B'(y)\, dy}{\int_{Y_\alpha} B'(y)\, dy} \tag{14}$$

where α is constant and Y_α is defined as

$$Y_\alpha = \{ y \in Y \mid B'(y) \geq \alpha \} \tag{15}$$

To eliminate the non-informative part under membership function where $B'(y) > \alpha$, the informative part or operative part of the membership function where $B'(y) > 0$, has to be parallel-shifted downward by the value of α according the formula

$$B^*(y) = \begin{cases} 0 & if B'(y) \leq \alpha \\ B'(y) - \alpha & if B'(y) > \alpha \end{cases} \tag{16}$$

In such case we may build a modified indexed center of gravity defuzzifier denoted by $MICOG_\alpha$ which may be expressed in the form:

$$y^* = \frac{\int_{Y_\alpha} y B^*(y)\, dy}{\int_{Y_\alpha} B^*(y)\, dy} = \frac{\int_{Y_\alpha} y (B'(y) - \alpha)\, dy}{\int_{Y_\alpha} (B'(y) - \alpha)\, dy} \tag{17}$$

Last modification can also be obtain considering ordinal COG defuzzifier on bounded difference, i.e.

$$y^* = COG[\max(0, B'(y) - \alpha)] \tag{18}$$

Other modifications are obtained when two cutting levels α_L (left-sided) and β_R (right-sided), ($\alpha_L > \beta_R$) for B'(y), appear. Denoting the cutting points for constant levels with B'(y), by y_α and y_β respectively, we get:

$$y^* = COG[B'(y) - \beta_R \mathbb{I}(y - y_\beta) - \alpha_L \mathbb{I}(y_\alpha - y)] \qquad (19)$$

denoted shortly $MICOG_{\alpha\beta}$, and

$$y^* = COG[B'(y) - \beta_R \mathbb{I}(y - y_\alpha) - \alpha_L \mathbb{I}(y_\alpha - y)] \qquad (20)$$

denoted $MICOG_{\alpha\alpha}$, where $\mathbb{I}(\cdot)$ denotes the Heaviside unit step pseudo-function. The interpretation of the above introduced defuzzifiers is obvious.

3. Denoting $y^{(k)}$ as the center of the k-th fuzzy set and τ_k as its height, the center average defuzzifier (CAD) or height method (HM) determines y* as

$$y^* = \frac{\sum_{k=1}^{K} \tau_k y^{(k)}}{\sum_{k=1}^{K} \tau_k} \qquad (21)$$

4. The maximum defuzzifier (MD) chooses the y* as the point in $\mathbb{Y}$ at which B'(y) achieves its maximum value i.e. defining the set of all points in $\mathbb{Y}$ at which B'(y) achieves its maximum value

$$Y(B') = \{y \in Y \mid B'(y) = \sup_{y \in Y} B'(y)\} \qquad (22)$$

the maximum defuzzifier determines y* as an arbitrary element in Y(B').

The mean of maximum (MOM) defuzzifier is defined as

$$y^* = \frac{\int_{Y(B')} y\, dy}{\int_{Y(B')} dy} \qquad (23)$$

where the integral denotes the conventional integration for the continuous part of Y(B') (or summation for the discrete part of Y(B')).

The modifications applied to defuzzifier 2, can also be applied to defuzzifiers 3 and 4 denoted shortly as $MISD_\alpha$, $MISD_{\alpha\alpha}$, $MISD_{\alpha\beta}$.

4. An equivalence of inference results using fuzzy implication interpretation and conjunctive interpretation of the if-then rules under defuzzification.

An output fuzzy set B'(y) obtained from inference system based on fuzzy implication interpretation of if-then rules is different from the resulting fuzzy set obtained from inference system based on conjunctive interpretation of fuzzy if-then rules. However in many applications crisp results are required instead of fuzzy ones. Hence, a question arises: whether it is possible to get the same or approximately the same crisp results from inference system when defuzzification is applied. The answer is positive under the respective circumstances. The point of departure of our considerations is the equality expressed in the form:

$$\delta_{mf_L}\left\{\sup_{\underline{x}\in X}\star_{T'}\left[\underline{A}'(\underline{x}),\ \begin{bmatrix}\bigwedge_{T\,k=1}^{K}\\ \sum_{k=1}^{K}\end{bmatrix} I(\underline{A}^{k}(\underline{x}),B^{k}(y))\,\right]\right\}=$$

$$\delta_{mf_R}\left\{\sup_{\underline{x}\in X}\star_{T'}\left[\underline{A}'(\underline{x}),\ \begin{bmatrix}\bigvee_{S\,k=1}^{K}\\ \sum_{k=1}^{K}\end{bmatrix} \underline{A}^{k}(\underline{x})\star_{T'}B^{k}(y)\,\right]\right\} \qquad (24)$$

The left-side part of that equality represents the defuzzified output of fuzzy implication based inference system whereas the right-side part represents an inference system based on Mamdani's composition (conjunctive interpretation of if-then rules).

The problem is to find such fuzzy implications (for the left-side part of the last equality), conjunctive operators (in the right-side of this equality), aggregation operations and defuzzification methods for both sides of the last equality in order to get the same crisp results. Generally, solving such a problem causes difficulties. One of the most important reasons is the different nature of fuzzy implication and conjunction (c.f. different truth table in classical logic). However, in special cases, under some assumptions, a pragmatic solution exists. To show such a solution let us assume for simplicity that the input fuzzy sets $\underline{A}'$ are singletons in $\underline{x}_0$ (in this particular case FATI is equivalent to FITA). The last equality can be rewritten in the simplified form:

$$\delta_{mf_L}\left\{\begin{bmatrix}\bigwedge_{T\,k=1}^{K}\\ \sum_{k=1}^{K}\end{bmatrix} I(\underline{A}^{k}(\underline{x}_0),B^{k}(y))\right\}=$$

$$\delta_{mf_R}\left\{\begin{bmatrix}\bigvee_{S\,k=1}^{K}\\ \sum_{k=1}^{K}\end{bmatrix} \underline{A}^{k}(\underline{x}_0)\star_{T'}B^{k}(y)\right\} \qquad (25)$$

Because of the differences in aggregation operations, we accept the same for both sides of the equality aggregation operation i.e. normalized arithmetic sum. Additionally, we assume different defuzzification methods for both sides of the last equality, e.g. MISD_α (modified indexed standard defuzzifier) for the left-side and SD (standard defuzzifier) for the right-side. For our purposes we will use as the left-side defuzzification method the method presented in the previous section named MICOG_α. For some fuzzy implications (e.g. Kleene-Dienes (max{1-a,b}), Lukasiewicz (min{1, 1-a+b}), Reichenbach (1-a+ab), Zadeh (max{1-a, min{a,b}}), Fodor (if a+b>1 then 1 else max{1-a,b}) and others) we have:

$$\alpha = \sum_{k=1}^{K}\alpha_k = \sum_{k=1}^{K}\overline{\underline{A}^{k}(\underline{x}_0)} = \sum_{k=1}^{K}(1-\tau^{k}) \qquad (26)$$

where τ^k stands for firing degree of k-th rule.

However, for some fuzzy implications (e.g. Gödel (if a≤b then 1 else b), Standard Sequence (if a≤b then 1 else 0), Goguen (if a≠0 then min{1,b/a} else 1) and others) α equals zero.

As the right side defuzzification method the well known COG method can be used. Taking into account the above mentioned assumptions and simplifications the last equality may be written as

$$MICOG_\alpha\left(\sum_{k=1}^{K} I(\underline{A}^k(\underline{x}_o), B^k(y))\right) = COG\left(\sum_{k=1}^{K} \underline{A}^k(\underline{x}_o) \star_{T'} B^k(y)\right) \tag{27}$$

It should be pointed out here that the modification in $MICOG_\alpha$ is responsible for elimination of the non-informative part of output membership function B'(y) (if such a part exists). As it was pointed out above.

The specific equivalence of inference results mentioned above can be seen straight on, if we take e.g. Reichenbach fuzzy implication under assumption that α is computed by means of the formula (26), as aggregation operation a normalized arithmetic sum is applied and the defuzzification method $MICOG_\alpha$ is represented by the formula (17) or (18). On the right side of the formula (27) algebraic product operation is taken in order to get Larsen's inference system with normalized arithmetic sum as aggregation operation and COG as defuzzification method. The above mentioned equivalence can also be describe by the chain of identities:

$$\begin{aligned} MICOG_\alpha\left[\sum_{k=1}^{K} (\overline{\underline{A}^k(\underline{x}_0)} + \underline{A}^k(\underline{x}_0) B^k(y))\right]_{Reich.} &= MICOG_\alpha\left[\sum_{k=1}^{K} (1-\tau_k) + \sum_{k=1}^{K} \tau_k B^k(y)\right]_{Reich.} \\ = MIHM_\alpha\left[\sum_{k=1}^{K} \alpha_k + \sum_{k=1}^{K} (1-\alpha_k) B^k(y)\right]_{Reich.} &= COG\left[\sum_{k=1}^{K} \tau_k B^k(y)\right]_{Larsen} \\ = HM\left[\sum_{k=1}^{K} \tau_k B^k(y)\right]_{Larsen} &= \left(\frac{\sum_{k=1}^{K} \tau_k y^{(k)}}{\sum_{k=1}^{K} \tau_k}\right)_{Larsen} \end{aligned} \tag{28}$$

When we consider the Lukasiewicz fuzzy implication under assumption that α is also computed using the formula (26), aggregation operation and $MICOG_\alpha$ are the same as in the previous case and on the right side of the formula (27), $*_T$ represents minimum, we get a well-known Mamdani's inference system with normalized arithmetic sum as aggregation operation and COG as defuzzification method. This equivalence can be described as follows:

$$\begin{aligned} MICOG_\alpha\left\{\sum_{k=1}^{K} [1 \wedge (\overline{\underline{A}^k(\underline{x}_0)} + B^k(y))]\right\}_{Luk.} &= MICOG_\alpha\left\{\sum_{k=1}^{K} [1 \wedge ((1-\tau_k) + B^k(y))\right\}_{Luk.} \\ = MICOG_\alpha\left\{\sum_{k=1}^{K} [1 \wedge (\alpha_k + B^k(y))]\right\}_{Luk.} &= COG\left\{\sum_{k=1}^{K} [1 \wedge (\alpha_k + B^k(y))] - \sum_{k=1}^{K} \alpha_k\right\}_{Mamdani} \\ = COG\left\{\sum_{k=1}^{K} [(1-\alpha_k) \wedge B^k(y)]\right\}_{Mamdani} &= COG\left\{\sum_{k=1}^{K} [\tau_k \wedge B^k(y)]\right\}_{Mamdani} \end{aligned} \tag{29}$$

Such formulated equivalence may be also shown using the generalized bounded difference c.f.[9]. In order to illustrate the considerations presented above some numerical examples will be discussed below.

5. Numerical examples

A fuzzy knowledge base presented in Fig 1 is taken as a basis of numerical calculation carried out here. Such knowledge base consisting of 9 fuzzy if-then rules may have practical meaning in many fields e.g. fuzzy control, fuzzy modeling, decision support systems and others. Each rule consists of two premises (e.g. error and change of error in the antecedent part of the rule and one conclusion (e.g. control) in consequent part. The fuzzy sets representing possible values of the respective linguistic variables are also shown in Fig. 1.

Using such a knowledge base we can geometrically show the essence of the above discussed equivalence. For all numerical examples the 'and' connective is considered to be an algebraic product. Considering Reichenbach implication with the normalized arithmetic sum as the aggregation operation and defuzzifier of $MICOG_\alpha$ type (α's are computed from the formula (26)), the equivalence of inference results with those obtained on the basis of Larsen's product is illustrated in Fig. 2. Taking into account the Lukasiewicz fuzzy implication with the same aggregation operation and defuzzifier of $MICOG_\alpha$ type for the same α's as in the previous case, the equivalence of inference results with inference results obtained on the basis of Mamdani's minimum is shown in Fig. 3. If we use the defuzzifier $MIHM_\alpha$ instead $MICOG_\alpha$ in the last case, then we get the same results as we got for Reichenbach implication. It means that the reduced (clipped) conclusion fuzzy sets are transformed into scaled conclusion fuzzy sets. Let us notice that the reduced fuzzy sets obtained using Lukasiewicz fuzzy implication include the scaled fuzzy sets.

In Fig. 4. the inference results obtained on the basis of Kleene-Dienes fuzzy implication are shown. However, the equivalence between the inference results obtained on the basis of this fuzzy implication and inference results obtained using a conjunction is not found but applying the defuzzifier $MIHM_\alpha$ instead of $MICOG_\alpha$ we get the same results as for Reichenbach fuzzy implication (see Fig.4). Because of the inclusion of Kleene-Dienes conclusion fuzzy sets in Reichenbach conclusion fuzzy sets the Kleene-Dienes conclusion fuzzy sets have to be respectively rescaled in order to get the last ones.

An analogous situation occurs if Fodor fuzzy implication is considered. The equivalence between the inference results obtained on the basis of this implication and results obtained by means of a conjunction is also not found. However, using $MIHM_\alpha$ defuzzifier instead of $MICOG_\alpha$ we get the same results as for Reichenbach fuzzy implication as well. Additionally it should be also pointed out that conclusion fuzzy sets obtained on the basis of Fodor fuzzy implication are not included in conclusion fuzzy sets obtained on the basis of Reichenbach fuzzy implication.

The rule base depicted in Fig. 1 was tested by means of complete set of singletons for the reason of fuzzy modeling of the function $z = f(x,y) = -(x + y) / 2$.

Four fuzzy implications (Reichenbach, Lukasiewicz, Kleene-Dienes and Fodor) were applied in inference system with the normalized arithmetic sum as an aggregation operation and defuzzifier of $MICOG_\alpha$ type, where α is computed from (26) as well.

For the above mentioned fuzzy implications the resulting surfaces are presented in Figs 6-9. These surfaces correspond to the surfaces obtained from the conjunctive based inference system. The quality indexes (λ_{MSE}, λ_{MAX}) characterizing the quality of inference system may be defined as follows:

$$\begin{aligned} \lambda_{MSE} &= \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} \left[I(x_i, y_j) - \frac{-(x_i + y_j)}{2} \right]^2 \\ \lambda_{MAX} &= \max_{i,j} \left| I(x_i, y_j) - \frac{-(x_i + y_j)}{2} \right| \end{aligned} \qquad (30)$$

The values of the above mentioned indexes are gathered in Table 1. It should be pointed out that both indexes for Reichenbach implication are equal to zero and for Lukasiewicz implication the values of indexes are the same as for Mamdani's minimum.

The programming tool applied to these investigations was FDSS Fuzzy-Flou and Fuzzy Logic toolbox for Matlab systems. Special procedures (m-files) concerning fuzzy implications and modified COG methods were written into the last mentioned toolbox.

Table 1.

Implication name	λ_{MSE}	λ_{MAX}
Kleene-Dienes	0,005428	0,1463
Reichenbach	0	0
Lukasiewicz	0,000524	0,0446
Fodor	0,000287	0,0418

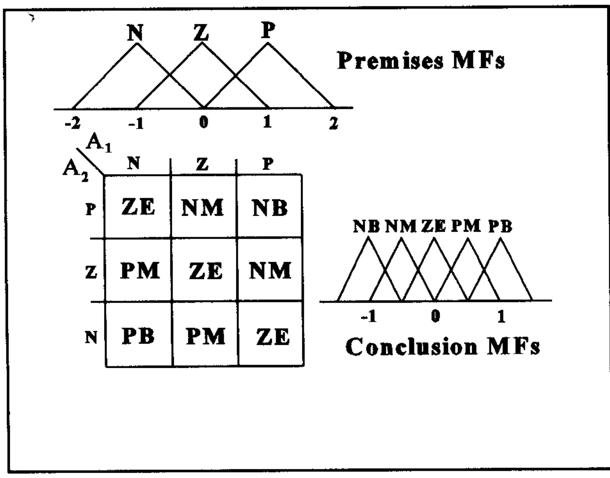

Fig. 1. Fuzzy knowledge base.

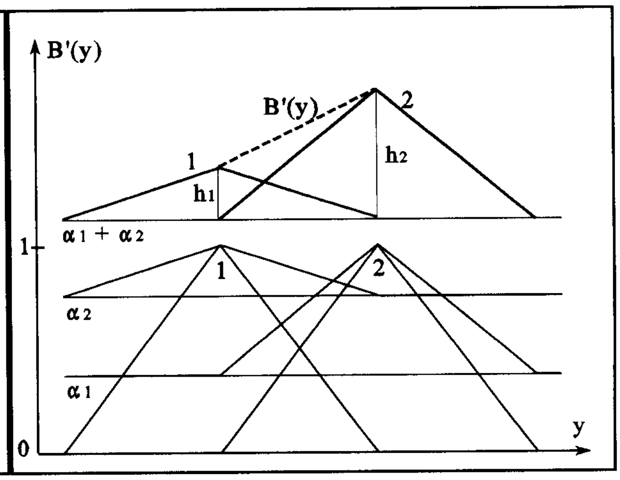

Fig. 2. An illustration of the equivalence of inference results (Reichenbach f.imp. versus Larsen's product).

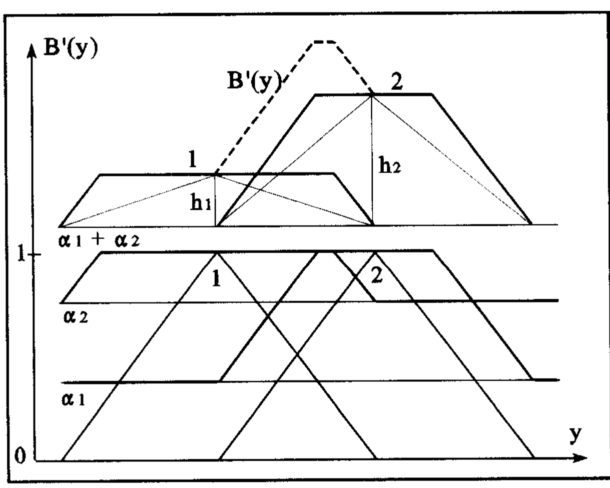

Fig. 3. An illustration of the equivalence of inference results (Lukasiewicz f.imp. versus Mamdani's minimum).

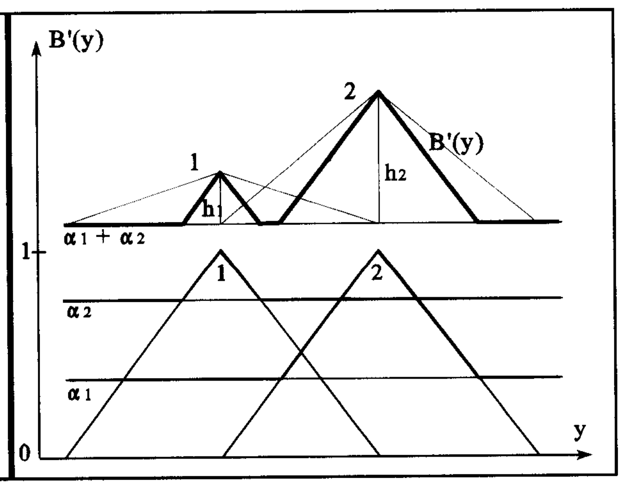

Fig. 4. An illustration of the equivalence of inference results (Kleene-Dienes f.imp. and Reichenbach f.imp.).

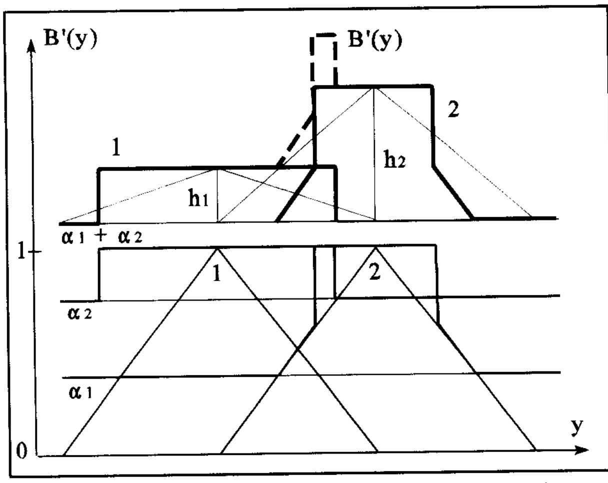

Fig. 5. An illustration of the equivalence of inference results (Fodor f.imp. and Reichenbach f.imp.).

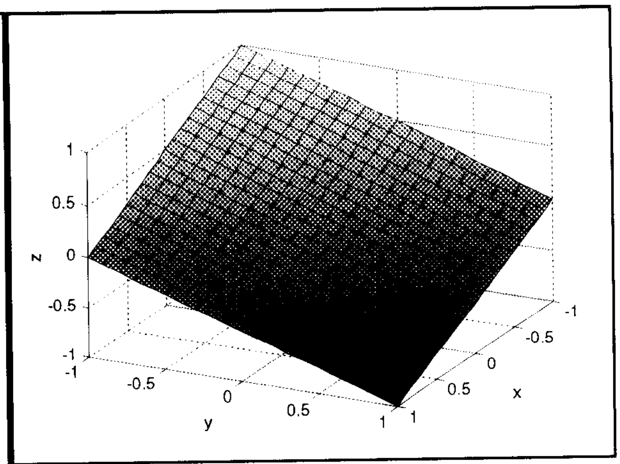

Fig. 6. Output surface for Reichenbach fuzzy implication

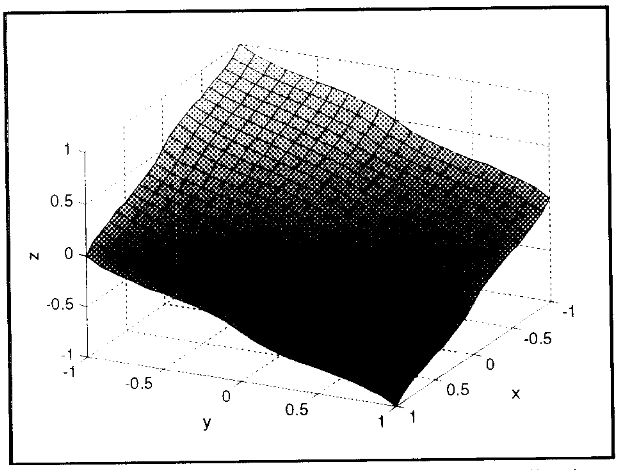

Fig. 7. Output surface for Lukasiewicz fuzzy implication.

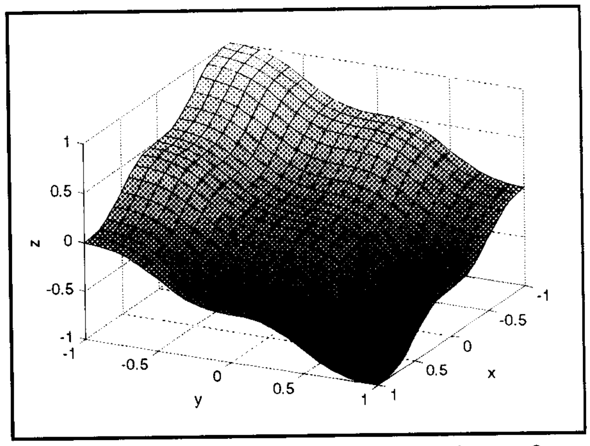

Fig. 8. Output surface for Kleene-Dienes fuzzy implication.

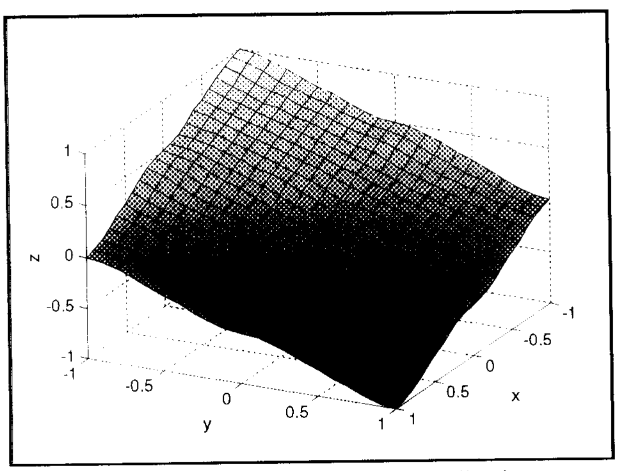

Fig. 9. Output surface for Fodor fuzzy implication

6. Conclusions

In this paper a specific type of equivalence of inference results using fuzzy implication (Reichenbach and Lukasiewicz) interpretation and the respective conjunctive (Larsen and Mamdani) interpretation of the if-then rules has been discussed. Such equivalence is important regarding the inference algorithm construction. The inference algorithms based on conjunctive operators in some cases seem to be faster, simpler and more exact than the algorithms of fuzzy implication based inference system. However, the interpretation of the fuzzy if-then rules based on fuzzy implications is sounder from logical point of view.

For future research, the equivalence mentioned above should be considered more deeply from mathematical point of view.

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