P-Evanescent Fuzzy Set-Valued Stochastic Processes

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Abstract: The concept of P-evanescent fuzzy set-valued stochastic process is proposed. A comparison between two fuzzy set-valued stochastic process is made. Further it is proved that a P-evanesent fuzzy set-valued stochastic process whose trajectories are (h) right continuous is a predictable fuzzy set-valued stochastic process.

Keywords: Fuzzy sets, set-valued, stochastic process, probability, predictable, P-evanescent.

1.Introduction

The theory of fuzzy set-valued stochastic process has been studied in [1] [2],. But there are still many properties of fuzzy set-valued stochastic process to be studied. This paper *purpose is to discuss a comparison between two fuzzy set-valued stochastic process and the relationship between P-evanescent fuzzy set-valued stochastic process and predictable fuzzy set-valued stochastic process. It is proven that a P-evanesent fuzzy set-valued stochastic process whose trajectories are (h) right continuous is a predictable fuzzy set-valued stochastic process.

For convenience, in section 2 we first introduce some basic definitions and results about fuzzy set-valued random variable, which may be found in [1], [2]. In section 3, we will give the main results, i.e. theorem 1 and theorem 2 and their proofs.

2. Basic notions of fuzzy set-valued stochastic process

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Let X be a n-dimension Euclidean space, $(\Omega, \Im, P)$ be a complete probability measure space, $\{\Im_t\}_{t\in R_+}$ be a family of monotone increasing sub-σ-fields of $\Im$, $\Im_{\infty-} = \bigvee_{t\in R_+} \Im_t$, $\Im_{\infty-} \subset \Im_\infty$ and $\Im_{0-} \subset \Im_0$, where $\Im_\infty$ and $\Im_{0-}$ are sub-σ-fields of $\Im$ in this paper.

Let $\widetilde{F}_0(X)$ be the family of all fuzzy sets $\widetilde{A}: X \to [0,1]$ with properties:

(1) $\widetilde{A}$ is upper semicontinuous,

(2) $\widetilde{A}$ is fuzzy covex,

(3) $\widetilde{A}_\alpha$ is compact, for every $\alpha \in (0,1]$,

where $\widetilde{A}_\alpha = \{x \in X: \widetilde{A}(x) \geq \alpha\}$ is the α-level set of $\widetilde{A}$.

If $\widetilde{A}, \widetilde{B} \in \widetilde{F}_0(X)$, define the distance between $\widetilde{A}$ and $\widetilde{B}$ by

$$d(\widetilde{A}, \widetilde{B}) = \sup_{\alpha>0} h(\widetilde{A}_\alpha, \widetilde{B}_\alpha)$$

where h denotes the Hausdorff distance.

$(\widetilde{F}_0(X), d)$ is a complete metric space.

A linear structure is defined in $(\widetilde{F}_0(X), d)$ by

$$(\widetilde{A} + \widetilde{B})(x) = \sup\{\alpha \in [0,1]: x \in (\widetilde{A}_\alpha + \widetilde{B}_\alpha)\}$$

$$(\lambda \widetilde{A})(x) = \begin{cases} \widetilde{A}(\lambda^{-1}x), & if \quad \lambda \neq 0, \\ 0, & if \quad \lambda = 0, \quad x \neq 0, \\ \sup_{y\in X} \widetilde{A}(y), & if \quad \lambda = 0, \quad x = 0, \end{cases}$$

for $\widetilde{A}, \widetilde{B} \in (\widetilde{F}_0(X), d)$, $\lambda \in R$. It is easy to prove that $(\widetilde{A} + \widetilde{B})_\alpha = \widetilde{A}_\alpha + \widetilde{B}_\alpha$, $(\lambda \widetilde{A})_\alpha = \lambda \widetilde{A}_\alpha$ for every $\alpha \in [0,1]$.

Definition 1. Let $\widetilde{F}: (\Omega, \Im) \to (\widetilde{F}_0(X), d)$ be a mapping from $(\Omega, \Im)$ to $(\widetilde{F}_0(X), d)$.

(1) $\widetilde{F}$ is called a fuzzy set-valued random variable, if

$$\{\omega : \sup_{y \in C} (\widetilde{F})(\omega)(x) \in B\} \in \Im$$

for any subset C of X and Borel's subset B of [0,1], i.e. B $\in \mathcal{B}$ ([0,1]).

(2) $\widetilde{F}$ is called $\Im$-level measurable if $\widetilde{F}_{\alpha}$ defined by $(\widetilde{F})_{\alpha}(\omega) = (\widetilde{F}(\omega))_{\alpha}$ for each $\omega \in \Omega$
is a random set for every $\alpha \in (0,1]$.

The following two properties are equivalent:
(1) $\widetilde{F}$ is a fuzzy set-valued random variable.
(2) $\widetilde{F}$ is $\Im$-level measurable
See theorem 1.5.1 in [2].

Definition.2. A family $\{\widetilde{F}_t\}_{t \in T}$ of fuzzy set-valued random variable is called a fuzzy set- valued stochastic process with pramatric set T.

Definition 3. (1) A subset Λ of $R_+ \times \Omega$ is called P-evanescent, if projection $\pi(\Lambda)$ of Λ on Ω is a P-zero probability set, i.e. $P(\pi(\Lambda))=0$.

(2) A set-valued stochastic process $F=\{F_t\}_{t \in R_+}$ is called evanescent if the set
$[(t,\omega): F_t(\omega) \neq \{0\}]$ is a P-evancscent set.

(3) A fuzzy set-valued stochastic process $\widetilde{F} = \{\widetilde{F}_t\}_{t \in R_+}$ is called a P-evanscent fuzzy set - valued stochastic process, if the set
$[(t,\omega): \widetilde{F}_t(\omega) \neq \{0\}]$is a P- evanscent set.

Definition 4. (1) Two set-valued stochastic processes $F=\{F_t\}_{t \in R_+}$ and $G=\{G_t\}_{t \in R_t}$ is called P-indistinguishable and denoted as F=G, if the set
$[[(t,\omega): F_t(\omega) \neq G_t(\omega)]$
is a P- evanscent set. G is said to be no less than F and denoted as $F \subseteq G$ if the set
$[(t,\omega): F_t(\omega) \not\subset G_t(\omega)]$
is a P- evanscent set.

(2) Two fuzzy set-valued stochastic processes $\widetilde{F} = \{\widetilde{F}_t\}_{t \in R_+}$ and $\widetilde{G} = \{\widetilde{G}_t\}_{t \in R_t}$ is called P-indistinguishable and denoted as $\widetilde{F} = \widetilde{G}$, if the set
$[[(t,\omega): \widetilde{F}_t(\omega) \neq \widetilde{G}_t(\omega)]$

is a P- evanscent set. $\widetilde{G}$ is said to be no less than $\widetilde{F}$ and denoted as $\widetilde{F} \subseteq \widetilde{G}$ if the set

$$[(t,\omega): \widetilde{F}_t(\omega) \not\subset \widetilde{G}_t(\omega)]$$

is a P- evanscent set.

3. Main theorems and their proofs

Theorem 1. Let $\widetilde{F} = \{\widetilde{F}_t\}_{t\in R_+}$ and $\widetilde{G} = \{\widetilde{G}_t\}_{t\in R_t}$ be two optional (resp. predictable) fuzzy set-valued stochastic processes. Then $\widetilde{G}$ is no less than $\widetilde{F}$ if $\widetilde{F}_T \subseteq \widetilde{G}_T$ a.s. for each bounded stopping time (resp. predictable time) T.

In order to prove theorem 1,we first give the following lemma.

Lemma 1. Let $F = \{F_t\}_{t\in R_+}$ and $G = \{G_t\}_{t\in R_t}$ be two optional (resp. predictable) set-valued stochastic processes.Then G is no less than F if $F_T \subseteq G_T$ a.s. for each bounded stopping time (resp. predictable time) T.

Proof. Since $F = \{F_t\}_{t\in R_+}$ and $G = \{G_t\}_{t\in R_t}$ are two optional (resp. predictable) set-valued stochastic processes. Suppose A= $\{(t,\omega): F_t(\omega) \not\subset G_t(\omega)\}$ is a P-nonevanescent set. But

$$A = \{(t,\omega): F_t(\omega) \not\subset G_t(\omega)\} = \{\bigcap_{n=1}^{\infty}[(t,\omega): f_t^{(n)}(\omega) \in G_t(\omega)]\}^c \in \mathcal{O}\ (\text{resp. } \mathcal{P}),$$

where $F_t(\omega) = cl\{f_t^{(n)}(\omega): n \geq 1\}$, $f_t^{(n)}$ is an optional (resp. predictable) selection of F_t. Then A is an optional (resp. predictable) set. Thus there exists a stopping time (resp. predictable time) such that $[[S]]=\{(t,\omega): S(\omega)=t\} \subset A$ and $P([S<\infty]) > 0$ by the section theorem . Suppose c is a constant number such that $P([S \leq c]) > 0$. Let $T = S \wedge c$. Then T is a bounded time (resp. predictable time) and $F_T(\omega) \not\subset G_T(\omega)$ for $\omega \in [S \leq c]$. This is in contradction with the assumption. Then A must be a P-evanescent set. Hence the set-valued stochastic process G is no less than the set- valued stochastic process F .

The proof of theorem 1. Since $\widetilde{F}$ and $\widetilde{G}$ are two optional (resp. predictable) fuzzy set-valued stochastic processes, then $\widetilde{F}_\alpha = \{(\widetilde{F}_t)_\alpha\}_{t\in R_+}$ and $\widetilde{G}_\alpha = \{(G_t)_\alpha\}_{t\in R_+}$ are two optional (resp. predictable) set-valued stochastic processes for each $\alpha \in (0,1]$.

Since $\widetilde{F}_T \subset \widetilde{G}_T$ a.s. for each bounded stopping time (resp. predictable time) T, then $(\widetilde{F}_T)_\alpha \subset (\widetilde{G}_T)_\alpha$ a.s. for each $\alpha \in (0,1]$. Hence $\widetilde{G}_\alpha$ is no less than $\widetilde{F}_\alpha$ for each $\alpha \in (0,1]$ by lemma 1. But

$$\widetilde{F}_t(\omega)(x) = \vee_{\alpha \in Q_0} [\alpha \wedge I_{(\widetilde{F}_t)_\alpha}(x)],\ \widetilde{G}_t(\omega)(x) = \vee_{\alpha \in Q_0} [\alpha \wedge I_{(\widetilde{G}_t)_\alpha}(x)]$$

where Q_0 is the set of all rational numbers in (0,1]. Then

$$\begin{aligned} & P\{\pi[(t,\omega): \widetilde{F}_t(\omega) \not\subset \widetilde{G}_t(\omega)]\} \\ &= P\{\pi(\bigcup_{\alpha \in Q_0} [(t,\omega):(\widetilde{F}_t)_\alpha(\omega) \not\subset (\widetilde{G}_t)_\alpha(\omega)])\} \\ &\le \sum_{\alpha \in Q_0} P\{\pi[(t,\omega):(\widetilde{F}_t)_\alpha(\omega) \not\subset (\widetilde{G}_t)_\alpha(\omega) \\ &= 0 \end{aligned}$$

Thus $\widetilde{G}$ is no less than $\widetilde{F}$.

Corollary 1. Let $\widetilde{F} = \{\widetilde{F}_t\}_{t\in R_+}$ and $\widetilde{G} = \{\widetilde{G}_t\}_{t\in R_+}$ be two optional (resp. predictable) fuzzy set-valued stochastic processes. Then $\widetilde{F}$ and $\widetilde{G}$ are indistingushable, if $\widetilde{F}_T = \widetilde{G}_T$ a.s. for each bounded stopping time (resp. predictable time) T.

Theorem 2. Let $\{\Im_t\}_{t\in \bar{R}_+ \cup \{0-\}}$ be complete, i.e. $\Im_{0-}$ includes all P-zero probability sets. Then each (d) right continuous P-evanescent fuzzy set-valued stochastic process $\widetilde{F} = \{\widetilde{F}_t\}_{t\in \bar{R}_+ \cup \{0-\}}$ is a predictable fuzzy set-valued stochastic process.

In order to prove theorem 2, we first give the following lemma 2.

Lemma 2. Let $\{\Im_t\}_{t\in \bar{R}_+ \cup \{0-\}}$ be complete. Then each (k) right continuous P-evanescent set-valued stochastic process $F = \{F_t\}_{t\in \bar{R}_+ \cup \{0-\}}$ is a predictable set-valued stochastic process, where (k) represents Kuratowski's sense.

Proof. Define $F^{(n)} = \{F_t^{(n)}\}_{t \in \bar{R}_+ \cup \{0-\}}$ as follows:

$$F_t^{(n)}(\omega) = F_0(\omega) I_{\{0\}}(t) + \sum_{k=0}^{\infty} F_{\frac{k+1}{n}}(\omega) I_{(\frac{k}{n}, \frac{k+1}{n}]}(t)$$

for each n ∈N, t ∈ R_+ and $\omega \in \Omega$.

Since F is a p-evanescent set-valued stochastic process, then $F_0(\omega) = \{0\}$ a.s. and $F_{\frac{k+1}{n}}(\omega) = \{0\}$ a.s.. Therefore F_0 and $F_{\frac{k+1}{n}}$ are both $\Im_{0-}$-measurable, since $\{\Im_t\}_{t \in \bar{R}_+ \cup \{0-\}}$ is complete. Because the predictable σ -field $\mathcal{P}=\sigma(\mathcal{C})$, where

$$\mathcal{C}=\{\{0\} \times A: A \in \Im_{0-}\} \cup \{(s,t] \times A: 0 < s < t, s,t \in Q_+, A \in \bigcup_{r<s} \Im_r\}$$

and Q_+ is the total of rational numbers in R_+. Then

$$\{(t,\omega): F_t^{(n)}(\omega) \cap G \neq \Phi\} = \{0\} \times [F_0 \cap G \neq \Phi] \cup (\bigcup_{k=0}^{\infty} (\frac{k}{n}, \frac{k+1}{n}] \times [F_{\frac{k+1}{n}} \cap G \neq \Phi]) \quad \mathcal{P},$$

where G is any open subset in X.. Then $F^{(n)}$ is a predictable set-valued stochastic process. Since $\{F_t\}_{t \in R_+}$ is (k) right continuous, then

$$\text{(k)} \lim_{n \to \infty} F_t^{(n)}(\omega) = F_t(\omega), \qquad \text{for } (t,\omega) \in R_+ \times \Omega.$$

Thus $F = \{F_t\}_{t \in R_+}$ is a predictable set-valued stochastic process.

The proof of theorem 2. Since $\widetilde{F} = \{F_t\}_{t \in R_+}$ is (d) right continuous P-evanescent fuzzy set-valued stochastic process, then $\widetilde{F}_\alpha = \{(\widetilde{F}_t)_\alpha\}_{t \in R_+}$ is a (h) right continuous compact set-valued stochastic process for each $\alpha \in (0,1]$, further $\widetilde{F}_\alpha = \{(\widetilde{F}_t)_\alpha\}_{t \in R_+}$ is a (k) right continuous compact set-valued stochastic process for each $\alpha \in (0,1]$ by theorem 1.5.33 in [4]. Since

$$0=\text{P}\, P\{\pi[(t,\omega): \widetilde{F}_t(\omega) \neq \{0\}]\} = P\{\pi(\bigcup_{\alpha \in Q_0} [(t,\omega):(\widetilde{F}_t)_\alpha(\omega) \neq \{0\}])\},$$

then $P\{\pi[(t,\omega):(\widetilde{F}_t)_\alpha(\omega) \neq \{0\}]\} = 0$ for each $\alpha \in Q_0$. But $\{(\widetilde{F}_t)_\alpha\}_{t \in R_+}$ is a predictable set-valued stochastic process for each $\alpha \in Q_0$ by lemma 2 and

$$(\widetilde{F}_t(\omega))(x) = \vee_{\alpha \in Q_0} (\alpha \wedge I_{(\widetilde{F}_t)_\alpha(\omega)}(x)).$$

Thus $\widetilde{F} = \{F_t\}_{t \in R_+}$ is a predictable fuzzy set-valued stochasic process.

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