Fuzzy Set-Valued Class(D) Stochastic Processes

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Abstract: The concepts of fuzzy set-valued class (D) stochastic process and measurable fuzzy set-valued stochastic process are given. It is proved that a fuzzy set-valued stochastic process whose almost all trajectories are (d) right continuous is measurable .The criterion of fuzzy set-valued class (D) stochastic process is got.

keywords: Fuzzy set, fuzzy set-valued random variable, fuzzy set-valued martingale, uniformly integrable family, fuzzy set-valued class (D) process.

1. Introduction

The theory of fuzzy set-valued stochastic process and fuzzy set-valued martingale have been studied in [1],[2],[3]. But there are still many properties of fuzzy set-valued martingale to be studied. This paper's pupose is to discuss the properties of fuzzy set-valued class (D) process and measurable fuzzy set-valued stochastic process e.t.. We will give the criterion of fuzzy set-valued class (D) stochastic process.

For convenience, in section 2 we first introduce some basic definitions and results of fuzzy set-valued stochastic process ,which may be found in [1],[2]. We will give the main theorems in this paper and their proofs in section 3.

2. Some basic concepts of fuzzy set-valued stochastic process and main results

Let X be a n-dimension Euclidean space, $(\Omega, \Im, P)$ be a complete probability measure space,

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$\{\Im_t\}_{t\in R_+}$ be a family of monotone increasing sub-σ-fields of $\Im$, $\Im_{\infty-} = \vee_{t\in R_+} \Im_t$, $\Im_{0-}$, $\Im_\infty$ be sub-σ-fields of $\Im$ and $\Im_{0-} \subset \Im_0$, $\Im_{-\infty} \subset \Im_\infty$ in this paper.

Let $\widetilde{F}_0(X)$ be the family of all fuzzy sets $\widetilde{A}: X \to [0,1]$ with properties:

(1) $\widetilde{A}$ is upper semicontinuous,

(2) $\widetilde{A}$ is fuzzy covex,

(3) $\widetilde{A}_\alpha$ is compact, for every $\alpha \in (0,1]$,

where $\widetilde{A}_\alpha = \{x \in X: \widetilde{A}(x) \geq \alpha\}$ is the α-level set of $\widetilde{A}$.

If $\widetilde{A}, \widetilde{B} \in \widetilde{F}_0(X)$, define the metric between $\widetilde{A}$ and $\widetilde{B}$ by

$$d(\widetilde{A}, \widetilde{B}) = \sup_{\alpha>0} h(\widetilde{A}_\alpha, \widetilde{B}_\alpha)$$

where h denotes the Hausdorff metric.

$(\widetilde{F}_0(X), d)$ is a complete metric space.

A linear structure is defined in $(\widetilde{F}_0(X), d)$ by

$$(\widetilde{A}+\widetilde{B})(x) = \sup\{\alpha \in [0,1]: x \in (\widetilde{A}_\alpha, +\widetilde{B}_\alpha)\}$$

$$(\lambda \widetilde{A})(x) = \begin{cases} \widetilde{A}(\lambda^{-1}x), & if \quad \lambda \neq 0, \\ 0, & if \quad \lambda = 0, \quad x \neq 0, \\ \sup_{y\in X} \widetilde{A}(y), & if \quad \lambda = 0, \quad x = 0, \end{cases}$$

for $\widetilde{A}, \widetilde{B} \in (\widetilde{F}_0(X), d)$, $\lambda \in R$. It is easy to prove that $(\widetilde{A}+\widetilde{B})_\alpha = \widetilde{A}_\alpha + \widetilde{B}_\alpha$, $(\lambda \widetilde{A})_\alpha = \lambda \widetilde{A}_\alpha$ for every $\alpha \in [0,1]$.

Definition 2.1. Let $\widetilde{F}: (\Omega, \Im) \to (\widetilde{F}_0(X), d)$ be a mapping from $(\Omega, \Im)$ to $(\widetilde{F}_0(X), d)$.

(1) $\widetilde{F}$ is called a fuzzy set-valued random variable, if

$$\{\omega: \sup_{y\in C} (\widetilde{F})(\omega)(x) \in B\} \in \Im$$

for any subset C of X and Borel's subset B of [0,1], i.e. B ∈**B** ([0,1]).

(2) $\widetilde{F}$ is called $\Im$-level measurable, if $\widetilde{F}_\alpha$ defined by $(\widetilde{F})_\alpha(\omega) = (\widetilde{F}(\omega))_\alpha$ for every

$\omega \in \Omega$ is a random set for every $\alpha \in (0,1]$.

Theorem 2.1 The following two propositions are equivalent:

(1) $\widetilde{F}$ is a fuzzy set-valued random variable.

(2) $\widetilde{F}$ is $\Im$-level measurable.

Proof. See theorem 1.5.1 in [1]

Definition 2.2. Let $\widetilde{F}:(\Omega,\Im)\rightarrow \widetilde{F}_0(X)$ be a fuzzy set-valued random variable, $\widetilde{F}$ is called to

be integrable bounded if there exists a nonnegative integrable function f such that

$\|\widetilde{F}_\alpha(\omega)\|$<f$(\omega)$ for each $\alpha \in(0,1]$ and $\omega \in \Omega$,

where $\|\widetilde{F}_\alpha(\omega)\|$=h{{0}, $\widetilde{F}_\alpha(\omega)$}

Define

$$(\int_\Omega \widetilde{F}dp)(x) = \vee_{\alpha\in[0,1]}(\alpha \wedge I_{\int_\Omega \widetilde{F}_\alpha dP}(x))$$

where $\int_\Omega \widetilde{F}_0 dP$=X and call $\int_\Omega \widetilde{F}dP$ to be integral of $\widetilde{F}$ on Ω.

Theorem 2.2. Let $\widetilde{F}:(\Omega,\Im)\rightarrow \widetilde{F}_0(X)$ be an integrable bounded fuzzy set-valued random variable, then $(\int_\Omega \widetilde{F}dP)_\alpha = \int_\Omega \widetilde{F}_\alpha dP$ for each $\alpha \in(0,1]$ and $\int_\Omega \widetilde{F}dP \neq \Phi$.

Proof: See theorem 6.5.3 in [1].

Theorem 2.3. Let $\widetilde{F}:(\Omega,\Im)\rightarrow \widetilde{F}_0(X)$ be an integrable bounded fuzzy set-valued random variable, then for each sub-σ-field $\Im_1$ of $\Im$, there exists an unique $\Im_1$-measurable fuzzy set-valued random variable $\widetilde{G}$ such that $\int_A \widetilde{F}dP = \int_A \widetilde{G}dP$ for each A $\in \Im_1$.

Proof: See theorem 6.5.4 [1].

Definition 2.3. Let $\widetilde{F}:(\Omega,\Im)\rightarrow \widetilde{F}_0(X)$ be an integrable bounded fuzzy set-valued random variable, $\Im_1$ be a sub-σ-field of $\Im$. $\widetilde{G}:(\Omega,\Im)\rightarrow \widetilde{F}_0(X)$ is called a conditional expectation

of $\widetilde{F}$ with respect to the sub-σ-field $\Im_1$ of $\Im$, and is denoted as E[$\widetilde{F}$ |$\Im_1$], if $\widetilde{G}$ is a $\Im_1$-
measurable integrable bounded fuzzy set-valued random variable satisfying the following condition:

$$\int_A \widetilde{F}dP = \int_A \widetilde{G}dP \quad \text{for each } A \in \Im_1 .$$

The conditional expectation of $\widetilde{F}$ with respect to the sub-σ-field $\Im_1$ of $\Im$ is existent and a.s. unique and $(E[\widetilde{F}|\Im_1](\omega))_\alpha = E[\widetilde{F}_\alpha|\Im_1](\omega)$ a.s. by theorem 2.2, theorem 2.3.

Definition 2.4. Let T be a set. The family of fuzzy random variables $\{\widetilde{F}_t\}_{t\in T}$ is called a fuzzy set- valued stochastic process with the set T of parameters.

Definition 2.5. Let $\{\Im_t\}_{t\in R_+}$ be a family of sub-σ-fields of $\Im$, $\{\widetilde{F}_t\}_{t\in R_+}$ be a $\{\Im_t\}_{t\in R_+}$-adapted integrably bounded fuzzy set-valued stochastic process, i.e. $\widetilde{F}_t$ is $\Im_t$-measurable and integrably bounded for each $t \in R_+$. $\{\widetilde{F}_t ; \Im_t\}_{t\in R_+}$ is said to be a fuzzy set-valued martingale (resp. supermartingale, submartingale) if

$$E[\widetilde{F}_t|\Im_s] = \widetilde{F}_s \text{ (resp. } \subseteq \widetilde{F}_s,\ \supseteq \widetilde{F}_s \text{) a.s. for } s<t,\ s, t \in R_+ .$$

Definition 2.6. A real function T defined on $(\Omega, \Im)$ is called a $\{\Im_t\}_{t\in R_+}$ stopping time if $\{\omega: T(\omega)<t\} \in \Im_t$ for each $t \in R_+$.

Definition 2.7. A fuzzy set-valued stochastic process $\{\widetilde{F}_t\}_{t\in R_+}$ is called measurable if

$$\{(t, \omega): \sup_{x\in C} \widetilde{F}_t(\omega)(x) \in B\} \in \mathbf{B}(R_+) \times \Im$$

for any closed C of X and any Borel's subset B of [0,1].

A $\{\Im_t\}_{t\in R_+}$-adapted fuzzy set-valued stochastic process $\{\widetilde{F}_t\}_{t\in R_+}$ is called progressive measurable if

$$\{(s, \omega) \in [0,t] \times \Omega : \sup_{x\in C} \widetilde{F}_s(\omega)(x) \in B\} \in \mathbf{B}([0,t]) \times \Im_t .$$

Definition 2.8. A family of fuzzy set-valued variables $\{\widetilde{F}_t : t \in T\}$ is said to be uniformly integrable if

$$\lim_{c\to+\infty} \int_{\{\omega:\|(\widetilde{F}_t)_\alpha(\omega)\|\geq c\}} \|(\widetilde{F}_t)_\alpha\| dp = 0 \text{ uniformly holds for } t \in T \text{ and } \alpha \in (0,1],$$

where $\|(\widetilde{F}_t)_\alpha(\omega)\| = h(\{0\}, (\widetilde{F}_t)_\alpha(\omega))$ is the Hausdorff metric between $\{0\}$ and $(\widetilde{F}_t)_\alpha(\omega)$.

Definition 2.9 . A $\{\Im_t\}_{t\in R_+}$ -adapted measurable fuzzy set-valued stochastic process $\{\widetilde{F}_t\}_{t\in R_+}$ is said to be a fuzzy set-valued class (D) process if $\{\widetilde{F}_T I_{[T<\infty]} : T\in\Re\}$ is uniformly integrable, where $\Re$ is the set of all $\{\Im_t\}_{t\in R_+}$ stopping times.

Respectively,when $\{F_t\}_{t\in R_+}$ is $\{\Im_t\}_{t\in R_+}$ -adapted set-valued stochastic process,we also define some concepts like those above concepts of fuzzy set-valued stochastic process.

3. Main theorems and their proofs

Theorem 3.1. If almost all trajectories of a fuzzy set-valued stochastic process $\{\widetilde{F}_t\}_{t\in R_+}$ are right continuous with respect to the metric d, then $\{\widetilde{F}_t\}_{t\in R_+}$ is a measurable fuzzy set-valued stochastic process.

Proof. Since almost all trajectories of $\{\widetilde{F}_t\}_{t\in R_+}$ are right continuous, there exists a $N\in\Im$, P(N)=0 such that

$$\lim_{\substack{t\to t_0\\ t>t_0}} d(\widetilde{F}_t(\omega),\widetilde{F}_{t_0}(\omega)) = \lim_{\substack{t\to t_0\\ t>t_0}}\{\sup_{\alpha\in(0,1]} h((\widetilde{F}_t)_\alpha(\omega),(\widetilde{F}_{t_0})_\alpha(\omega))\} = 0$$

for $t_0\in R_+$, $\omega\notin N$. Then

$$\lim_{\substack{t\to t_0\\ t>t_0}} h((\widetilde{F}_t)_\alpha(\omega),(\widetilde{F}_{t_0})_\alpha(\omega)) = \lim_{\substack{t\to t_0\\ t>t_0}}\{\sup_{x\in X}\left|d_\varepsilon(x,(\widetilde{F}_t)_\alpha(\omega)) - d_\varepsilon(x,(\widetilde{F}_{t_0})_\alpha(\omega))\right|\} = 0$$

for each $\alpha\in(0,1]$, where d_ε is the metric in Euclid space X. Hence

$$\lim_{\substack{t\to t_0\\ t>t_0}} d_\varepsilon(x,(\widetilde{F}_t)_\alpha(\omega)) = d_\varepsilon(x,(\widetilde{F}_{t_0})_\alpha(\omega))$$

for each $x\in X$. Therefore $\{d_\varepsilon(x,(\widetilde{F}_t)_\alpha(\omega))\}_{t\in R_+}$ is a stochastic process whose almost all trajectories are continuous for each $\alpha\in(0,1]$. Thus $\{d_\varepsilon(x,(\widetilde{F}_t)_\alpha(\omega))\}_{t\in R_+}$ is a measurable stochastic process for each $\alpha\in(0,1]$ and futher $\{(\widetilde{F}_t)_\alpha\}_{t\in R_+}$ is a measurable set-valued stochastic process for each $\alpha\in(0,1]$. Thereby $\{\widetilde{F}_t\}_{t\in R_+}$ is a measurable fuzzy set-valued stochastic process .

Theorem 3.2. Let $\{\widetilde{F}_t\}_{t\in R_+}$ be a $\{\Im_t\}_{t\in R_+}$ -adapted fuzzy set-valued stochastic process whose all trajectories are right continuous with respect to the metric d. Then $\{\widetilde{F}_t\}_{t\in R_+}$ is a progressive measurable fuzzy set-valued stochastic process.

In order to prove this theorem, firstly we give the following lemma.

Lemma 3.1. Let $\{F_t\}_{t\in R_+}$ be a $\{\Im_t\}_{t\in R_+}$-adapted set-valued stochastic process whose trajectories are right continuous with respect to the metric h. Let F_t be integrably bounded for each $t \in R_+$. Then $\{F_t\}_{t\in R_+}$ is a progressive measurable set-valued stochastic process.

Proof. Define a sequence of set-valued stochastic processes $\{F_s^{(n)}\}_{s\in[0,t]}$ as the following:

$$F_s^{(n)}(\omega) = F_0(\omega) I_{[S=0]} + \sum_{k=1}^{2^n} F_{\frac{kt}{2^n}}(\omega) I_{[\frac{(k-1)t}{2^n},\frac{kt}{2^n}]}(s).$$

Then

$$\{(s,\omega) \in [0,t]\times\Omega : F_s^{(n)}(\omega)\bigcap G \neq \Phi\}$$
$$= \{0\}\times[\omega: F_0(\omega)\bigcap G \neq \Phi]\bigcup\{\bigcup_{k=1}^{2^n}[\frac{(k-1)t}{2^n},\frac{kt}{2^n}]\times[\omega: F_{\frac{kt}{2^n}}(\omega)\bigcap G \neq \Phi]\} \in \mathbf{B}([0,t])\times\Im_t$$

for each open subset G of X.. Thus

$$\{(s,\omega) \in [0,t]\times\Omega : d_\varepsilon(x, F_s^{(n)}(\omega)) < r\} \in \mathbf{B}([0,t])\times\Im_t$$

for each $r \in R$ and $x \in X$ by corollary 1.1.2 in [1].

Since $\lim_{n\to\infty} h(F_s^{(n)}(\omega), F_s(\omega)) = \lim_{n\to\infty}\{\sup_{x\in X}|d_\varepsilon(x, F_s^{(n)}(\omega)) - d_\varepsilon(x, F_s(\omega))|\}=0$, then $\lim_{n\to\infty} d_\varepsilon(x, F_s^{(n)}(\omega)) = d_\varepsilon(x, F_s(\omega))$. Hence $\{(s,\omega) \in [0,t]\times\Omega : d_\varepsilon(x, F_s(\omega)) < r\} \in \mathbf{B}([0,t])\times\Im_t$ for each $r \in R$ and $x \in X$. Thus $\{(s,\omega): F_s(\omega)\bigcap G \neq \Phi\} \in \mathbf{B}([0,t])\times\Im_t$ by corollary1.1.2 in [1]. Therefore $\{F_t\}_{t\in R_+}$ is progressive measurable.

Proof of theorem 3.2: Since $\{\widetilde{F}_t\}_{t\in R_+}$ is a $\{\Im_t\}_{t\in R_+}$-adapted fuzzy set-valued stochastic process whose trajectories are right continuous with respect to the metric d, for each $\alpha \in (0,1]$, $\{(\widetilde{F}_t)_\alpha\}_{t\in R_+}$ is a $\{\Im_t\}_{t\in R_+}$-adapted set-valued stochastic process whose trajectories are right continuous with respect to the Hausdorff metric h.. Then $\{(\widetilde{F}_t)_\alpha\}_{t\in R_+}$ is a progressive measurable set-valued stochastic process for each $\alpha \in (0,1]$ by lemma 3.1. Therefore $\{\widetilde{F}_t\}_{t\in R_+}$ is a progressive measurable fuzzy set-valued stochastic process by theorem 2.1.

Theorem 3.3. Suppose $\{\widetilde{F}_t\}_{t\in \overline{R}_+}$ is a $\{\Im_t\}_{t\in \overline{R}_+}$-adapted progressive fuzzy set-valued stochastic process whose trajectories are left continuous with respect to the metric d, then $\widetilde{F}_T I_{[T<\infty]}$ and

$\widetilde{F}_T = \widetilde{F}_T I_{[T<\infty]} + \widetilde{F}_\infty I_{[T=\infty]}$ are $\Im_T$ -measurable for every $\{\Im_t\}_{t\in\bar{R}_+}$ -stopping time T, where

$\Im_T = \{ A \in \Im_\infty : A\bigcap[T \le t] \in \Im_t$ for each $t \in R_+ \}$.

In order to prove theorem 3.3 we first give thefollowing lemma, i.e., lemma 3.2.

Lemma 3.2. Suppose $\{ F_t \}_{t\in\bar{R}_+}$ is a $\{\Im_t\}_{t\in\bar{R}_+}$ -adapted progressive set-valued stochastic process whose trajectories are left continuous with respect to the Hausdorff metric h, then $F_T I_{[T<\infty]}$ and $F_T = F_T I_{[T<\infty]} + F_\infty I_{[T=\infty]}$ are $\Im_T$ -measurable for every $\{\Im_t\}_{t\in\bar{R}_+}$ -stopping time T.

Proof. For each $\{\Im_t\}_{t\in\bar{R}_+}$ -stopping time T, since $T\wedge t$ is $\Im_t$ -measurable for each $t \in \bar{R}_+$, and $F_{T\wedge t}$ can be considered as the composition of the measurable mapping $\omega \mapsto (T(\omega)\wedge t, \omega)$ from $(\Omega, \Im_t)$ to $([0,t]\times\Omega, \mathbf{B}([0,t])\times\Im_t)$ and the measurable set-valued mapping $(s,\omega) \mapsto F_s(\omega)$ from $([0,t]\times\Omega, \mathbf{B}([0,t])\times\Im_t)$ to X. Then $F_{T\wedge t}$ is $\Im_t$ -measurable. further

$$[F_T I_{[T<\infty]} \bigcap G \ne \Phi]\bigcap[T \le t] = [F_{T\wedge t} \bigcap G \ne \Phi]\bigcap[T \le t] \in \Im_t$$

for any open subset G of X and

$$F_T I_{[T<\infty]} = F_T \lim_{n\to\infty} I_{[T<n]} = \lim_{n\to\infty} F_{T\wedge n} I_{[T<n]}$$

with respect to the Hausdorff metric h by the continuity of $\{F_t\}_{t\in\bar{R}_+}$. Then $F_T I_{[T<\infty]}$ is $\Im_\infty$-measurable. Consequently $[F_T I_{[T<\infty]} \bigcap G \ne \Phi] \in \Im_T$ for any open subset G of X, i.e. $F_T I_{[T<\infty]}$ is $\Im_T$ -measurable for each $\{\Im_t\}_{t\in\bar{R}_+}$ -stopping T.

It can be proved easyly by similar method that $F_T = F_T I_{[T<\infty]} + F_\infty I_{[T=\infty]}$ is $\Im_T$ -measurable for each $\{\Im_t\}_{t\in\bar{R}_+}$ -stopping T.

Proof of theorem 3.3. Since for each $\alpha \in (0,1]$, $\{(F_t)_\alpha\}_{t\in\bar{R}_+}$ is a $\{\Im_t\}_{t\in\bar{R}_+}$ -adapted progressive set-valued stochastic process, and its trajectories are left continuous with respect to the Hausdorff metric h, then $(\widetilde{F}_T)_\alpha I_{[T<\infty]}$ and $(\widetilde{F}_T)_\alpha = (\widetilde{F}_T)_\alpha I_{[T<\infty]} + (\widetilde{F}_\infty)_\alpha I_{[T=\infty]}$ are both $\Im_T$ -measurable for each $\{\Im_t\}_{t\in\bar{R}_+}$ -stopping T by lemma 3.2. Thus $\widetilde{F}_T I_{[T<\infty]}$ and $\widetilde{F}_T = \widetilde{F}_T I_{[T<\infty]} + \widetilde{F}_\infty I_{[T=\infty]}$ are both $\Im_T$ -measurable for each $\{\Im_t\}_{t\in\bar{R}_+}$ -stopping time T by theorem 2.1.

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Theorem 3.4. Let $\{\widetilde{F}_t;\Im_t\}_{t\in\overline{R}_+}$ be a fuzzy set-valued submartingale and its almost all trajectories be right continuous with respect to the metric d, then $\{\widetilde{F}_t\}_{t\in\overline{R}_+}$ is a fuzzy set-valued class (D) stochastic process.

Proof. By asumption and theorem 3.1, $\{\widetilde{F}_t\}_{t\in\overline{R}_+}$ is a measurable fuzzy set-valued stochastic process and $\widetilde{F}_\infty$ is integrable bounded. Then $\{E[\widetilde{F}_\infty|\Im_T]\}_{T\in\overline{\Re}}$ is uniformly integrable by theorem 4.3 in [3], where $\overline{\Re}$ is the set of all $\{\Im_t\}_{t\in\overline{R}_+}$ -stopping times and $E[\widetilde{F}_\infty|\Im_T]\supseteq\widetilde{F}_T$ a.s. by theorem 3.4 in [2]. Therefore $\{\widetilde{F}_T\}_{T\in\overline{\Re}}$ is uniformly integrable by theorem 4.1 in[3]. Since $\|(\widetilde{F}_T I_{[T<\infty]})_\alpha\|\le\|(\widetilde{F}_T)_\alpha\|$ for $\alpha\in(0,1]$, then $\{(\widetilde{F}_T I_{[T<\infty]})_\alpha : T\in\overline{\Re},\ \alpha\in(0,1]\}$ is uniformly integrable. Thus $\{\widetilde{F}_T I_{[T<\infty]}:T\in\overline{\Re}\}$ is uniformly integrable. Further $\{\widetilde{F}_t\}_{t\in\overline{R}_+}$ is a fuzzy set-valued class (D) stochastic process.

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