# THE LAW OF LARGE NUMBERS AND T-STABILITY FUZZY NUMBERS

R.R.Salakhutdinov, NC, USA
R.Z.Salakhutdinov, Scientific Industrial Union "Cybernetics" Tashkent Uzbekistan.

Introdused the concept T-stable membership function. When T-sum of fuzzy nambers with common T-stable membership function obey (or not obey) the law of large numbers for fuzzy nambers is representing.
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1. Introduction. Let $\xi_1, \xi_2, \ldots$ is a sequence of fuzzy numbers with common triangular membership function $\xi_i(x;m,\alpha) = 1 - |x-m_i|/\alpha$, if $m_i-\alpha \leqslant x \leqslant m_i+\alpha$; otherwise $\xi_i = 0$. $\alpha$ - is its width; $m_i$ - is its modal values; $(\alpha>0, -\infty < m_i < \infty)$.

By $(\xi_1+\xi_2)_T$ denoted T-sum two fuzzy numbers $\xi_1$, $\xi_2$ and its membership function defined as

$$(\xi_1+\xi_2)_{T}(z) = \sup_{x+y=z} T(\xi_1(x), \xi_2(y)),$$

Here T is t-norm. As the examples of t-norms are Hamacher's $(H_r)$ and Dombi's $(D_q)$ operators [1,2]:

$$H_r(u,v) = \frac{uv}{r+(1-r)(u+v-uv)}, \quad r>0 \qquad D_q(u,v) = \left\{1+\left[\left(\frac{1-u}{u}\right)^q+\left(\frac{1-v}{v}\right)^q\right]^{1/q}\right\}^{-1}, \quad q>0$$

Obviously, that for the tasks of applicable character it is interesting to study the behavior of the T-sum of fuzzy nambers $S_n = ((\xi_1+\xi_2+\ldots+\xi_n)/n)_T$ when $n \to \infty$.

In R.Fuller's paper [2] is shown that if t-norm T is weaker than $H_0$ - Hamacher's operator, then is fairly the law of large numbers for symmetric triangular fuzzy numbers

$$\lim_{n\to\infty} \mathrm{Nes}\left[M_n-\beta \leqslant \left(\frac{\xi_1+\xi_2+\ldots+\xi_n}{n}\right)_T \leqslant M_n+\beta\right] = 1, \qquad (1)$$

$$M_n = \frac{m_1+m_2+\ldots+m_n}{n}$$

Here Nes ($a \ll \xi \ll b$) is interpreted as the grade of necessity of the statement " [a,b] contains the values of $\xi$ ".

Our purpose is to investigate under which conditions for a membership function and T-summation will obey or not obey the law of large numbers.

2. Results.

A. Let's formulate the definition of T-stable membership function.

Definition 1. Let $\xi_1$, $\xi_2$ are fuzzy numbers with common membership function $\xi(x; m,\{a^*\})$ and $\xi(x; k,\{b^*\})$. If membership function of the T-sum $(\xi_1+\xi_2)_T$ has the same membership function $\xi(x;h,\{c^*\})$, then membership function $\xi$ is T-stable.

Here m, k, h - are corespondingly modal values; $a^*$, $b^*$, $c^*$ are its coresponding vectors of the widths. Notice, for example, that in case of symmetric triangular numbers: $\{a^*\}=\alpha$, and in case L-R fuzzy numbers Dubois & Prade: $\{a^*\} = \{\alpha,\delta\}$.

If $T(u,v) = \min(u,v)$, then we have min-stable.

If $T(u,v) = u*v$, then we have Tp-stable.

If $T(u,v) = \max(0,u+v-1)$, then we have Tm-stable, ets.

Example 1. A symmetrical triangular membership function is min-stable. Let $\xi(x; m_1,\alpha_1)$ and $\xi(x; m_2,\alpha_2)$ are membership functions of triangular form, then from [3]: $\xi(x; m_1+m_2, \alpha_1+\alpha_2)$ is membership function of $(\xi_1+\xi_2)_T$.

Example 2. Gauchy's memebership function

$$\xi_i(x) = \xi(x; m_i,\alpha) = \exp\left(-\frac{(x-m_i)^2}{\alpha^2}\right) \qquad \text{is Tp-stable.}$$

Indeed, $(\xi_1+\xi_2)_{Tp}(z) = \sup_{x+y=z} T_p(\xi_1(x),\xi_2(y)) =$

$$= \sup_{x+y=z} \{\exp(-(x-m_1)/\alpha)^2 * \exp(-(y-m_2)/\alpha)^2\} =$$

$$= \sup_{x} \exp\{-(z-(m_1+m_2))/\alpha^2 + 2(x-m_1)\ (z-x-m_2)/\alpha^2\} =$$

$$= \exp\{\ [-(z-(m_1+m_2)) + 2(z-(m_1+m_2))/4]/\alpha^2\} = \xi(z;m_1+m_2,\alpha\sqrt{2}).$$

Example 3. Let we have $\xi_i(x) = \xi_i(x;\ 0,\ 1) = 1 - x^2$, $-1 \leqslant x \leqslant 1$; else $\xi_i(x) = 0$. Then

$$(\xi_1+\xi_2)_{Tm}(z) = \sup_{x+y=z} T_m(\xi_1(x),\xi_2(y)) = \sup_{x+y=z} (1-x^2 + 1-y^2 - 1) =$$

$$= \sup_{x+y=z} (1 - (z^2-2xy)) = 1 - (z^2 - 2z^2/4) = 1 - z^2/2.$$

Hense, we have here Tm-stable membership function.

Example 4. Assuming the following conditions for membership function [4]:

1. $\xi(x)$ has a modal value m, $\xi(m)=1$;
2. $\xi(x)$ is symmetric around m, $\xi(x-m) = \xi(m-x)$;
3. $\xi(x)$ is an increasing in the interval $(-\infty,\ m)$;

Let $\xi_1$, $\xi_2$ be fuzzy numbers with common membership function $\xi(x;m,\{a^*\})$,that satisfies conditions 1-3, then $\xi$ is min-stable.

Indeed, from Rao & Rashed theorem's [4] it follows that min-sum of $(\xi_1+\xi_2)_{min}$ has the same membership function $\xi$ with a corresponding parameters.

Note. Membership function is interpreting as possibility distribution [1]. Due to this, could be introduce the notion of T-stable distribution of possibilities.

B. Further the grade of possibility of the statement "[a,b] contains the values of $\xi$" is defined by $Pos(a \ll \xi \ll b) = \sup_{a<x<b} \xi(x)$, [1,2].

From this it follows that, for a fuzzy numbers $\xi$ with membership function $\xi(x) = \xi(x;\ m,\ \alpha)$, we can write

$$Nes\ (a \ll \xi \ll b) = 1-Pos(\xi<a,\xi>b) = 1- \sup_{x<a;\,x>b} \xi(x) = 1-\max(\xi(a),\ \xi(b)).$$

In particular, for membership function that satisfyes conditions 1-3 from the example 4, we have

$$Nes\ (m-\beta \ll \xi \ll m+\beta) = 1 - \sup_{\substack{x<m-\beta \\ x>m+\beta}} \xi(x) = 1 - \xi(\beta/\alpha), \quad \beta<\alpha \qquad (2)$$

Thus, substituting in the relationship (2) instead $\xi$ (resp: $\xi(x)$), $S_n = ((\xi_1+\xi_2+...+\xi_n)/n)_T$ (resp: its membership function), could be conclude is the law of large numbers fair or not.

Some considerations we are illustrating by table.

| Membership function $\xi$ | T-norm | Membership function $S_n$ | T-stability | Law of larg numbers |
|---|---|---|---|---|
| Symmetric triangular | $H_0$ Hamacher | $A/B$ <br> $A = \xi(x;\ M_n,\alpha)$ <br> $B = n-(n-1)\xi(x;\ M_n,\alpha)$ | no | yes |
| Symmetric triangular | $D_q$ Dombi | $A/B$ <br> $A = \xi(x;\ M_n\ \alpha)$, $B =$ <br> $= n-(n^{1/q}-1)\xi(x;\ M_n,\alpha)$ <br> $q>0$ | no | yes |
| Symmetric triangular | Tp | $\left\{\xi(x;\ M_n,\alpha)\right\}^n$ | no | yes |
| Gauchy | Tp | $\left\{\xi(x;\ M_n,\alpha\sqrt{2}\right\}^n$ | no | yes |
| Symmetric triangular | min | $\xi(x;\ M_n,\alpha)$ | yes | no |

Note. Taking into consideration that $H_\infty \ll H_1 \ll \ldots \ll H_0 = D_1 \ll \ldots\ D_2 \ll \ldots \ll D_\infty = \min$, we can improve of R.Fuller's result (1): If $T < \min$, then the law of large numbers for symmetric triangular fuzzy numbers is fair [5].

Possible, that the law of large numbers is true, only for not T-stable membership function.

References

[1] Dubois D., Prade H. Fuzzy sets and systems:theory and applications .-N.Y: Acad.Press, 1980, 340 p.

[2] Fuller R. The law of large numbers for fuzzy numbers, BUSEFAL 40 (1989), 25-32.

[3] Nahmias S. Fuzzy variables, Fuzzy Sets and Systems 1 (1978), 299-309

[4] Rao M.B., Rashed A. Some comments on fuzzy variables, Fuzzy Sets and Systems, 6 (1981), 285-292

[5] Salakhutdinov R.Z. On the law of large numbers for fuzzy numbers (to appear).