On thd local minimum value and the global minimum value of the convex fuzzy valued functions

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Abstract: In this paper, on the basis of the convex fuzzy valued mappings given by Nanda [1], the convex fuzzy valued functions are defined in the Euclidean space R^n, and the result which their local minimum value and global minimum value is equivalent is obtained.

Keywords: fuzzy numbers; convex fuzzy vaued functions; local minimum point; global minimum point.

1 Introduction

The concept of the convex fuzzy sets were originally introduced by L. A. Zadeh [7]. Subsequently a lot of scholars did a great deal of work at the aspects of their theories and applications. Some properties of the convex fuzzy sets were studied and given by Lowen [2], Katsaras and Liu [3], Brown [4]. The concept of the convex fuzzy mapping has first been introduced and some results including some applications to nondifferentiable optimization have been investigated by Nanda [1]. In this paper, We will study the equivalence problem of the local minimum value and the global minimum value of the convex fuzzy valued functions.

2 Preliminaries

Let $[H] = \{\bar{a} = [a^-, a^+] : a^- \leqslant a^+, a^-, a^+ \in R\}$ denote the set of all interval numbers on the real number field R.

Let $\bar{a}, \bar{b}, \bar{c} \in [H]$, where $\bar{a} = [a^-, a^+]$, $\bar{b} = [b^-, b^+]$, $\bar{c} = [c^-, c^+]$.

We define

$(1)\ \bar{a} = \bar{b}$ iff $a^- = b^-, a^+ = b^+$

$(2)\ \bar{a} \leqslant \bar{b}$ iff $a^- \leqslant b^-, a^+ \leqslant b^+$

$(3)\bar{a} < \bar{b}$ iff $\bar{a} \leqslant \bar{b}$ *and* $\bar{a} \neq \bar{b}$

$(4)\bar{a} + \bar{b} = \bar{c}$ iff $a^- + b^- = c^-, a^+ + b^+ = c^+$

$(5)k\bar{a} = [ka^-, ka^+], k \geqslant 0$

For $\bar{a}, \bar{b} \in [H]$, the distance is defined as

$$d(\bar{a}, \bar{b}) = \max\{|a^- - b^-|, |a^+ - b^+|\}.$$

Obviously, $([H], d)$ is a metric space, where d is an Hausdorff distance, and $([H], \leqslant)$ constitutes a partial order set.

Definition 2.1 Let the mapping $A: R \to [0,1]$. Then A is called a fuzzy number on R, if the following conditions are satisfied.

(1) A is normal, i.e., there exists $x_0 \in R$ such that $A(x_0) = 1$.

(2) A is fuzzy convex, i.e., for arbitrary $x, y \in R, \lambda \in [0,1]$,

$$A(\lambda x + (1-\lambda)y) \geqslant A(x) \wedge A(y) \text{ holds}.$$

(3) A is upper semi – continuous

(4) The support $\mathrm{Supp}A = \{x \in R : A(x) > 0\}$, its closure $\overline{\mathrm{supp}A}$ is compact.

Denote the family of all fuzzy numbers on R as $F^*(R)$.

Lemma 2.1 If $A \in F^*(R)$. Then for any $\alpha \in (0,1]$, its level set $A_\alpha = \{x \in R : A(x) \geqslant \alpha\}$ is a closed interval.

Let the mapping $\bar{\rho}: F^*(R) \times F^*(R) \to R$

$$\bar{\rho}(A, B) = \sup_{0 \leqslant \alpha \leqslant 1} d(A_\alpha, B_\alpha)$$

We are easy to know that $\bar{\rho}$ is also a metric on $F^*(R)$.

Let $A, B, C \in F^*(R)$, define

$$A \leqslant B \text{ iff } A_\alpha \leqslant B_\alpha. \text{ for arbitrary } \alpha \in [0,1].$$

$$A + B = C \text{ iff } A_\alpha + B_\alpha = C_\alpha. \text{ for arbitrary } \alpha \in [0,1]$$

In particularly, for $A \in F^*(R), k \geqslant 0, x \in R$, We define

$$(kA)(x) = \begin{cases} A(\frac{x}{k}) & if\ k \neq 0 \\ 1 & for\ x = 0\ if\ k = 0 \\ 0 & for\ x \neq 0\ if\ k = 0 \end{cases}$$

Proposition 2.1 If $A \in F^*(R), k > 0$, Then $(kA)_\alpha = kA_\alpha$, *for* any $\alpha \in (0,1]$

Proof for arbitrary $x \in (kA)_\alpha$ iff $(kA)(x) \geqslant \alpha$. i.e, $A(\frac{x}{k}) \geqslant \alpha$ iff $\frac{x}{k} \in A_\alpha$ or $x \in kA_\alpha$.

Proposition 2.2 If $A, B \in F^*(R)$ and $A \leqslant B, k > 0$. Then $kA \leqslant kB$.

Proposition 2.3 If $A \in F^*(R), k_1 > 0,\ k_2 > 0$. Then $(k_1 + k_2)A = k_1A + k_2A$.

3. Main results

In the course of applications, as if in the classical real valued functions, we are interested in the global properties of the extremum, i.e., in the whole field of definitions Ω and the local neighbourhood $U\ (x_0, \delta)$ regarding x_0 as the centre, σ as the radius, whether we have $f(x) \geqslant f(x_o)$, for all $x \in \Omega$ iff $f(x) \geqslant f(x_0)$, for all $X \in U(x_0, \delta)$? In the ordinary case, it is not certain. But as for the convex fuzzy ualued functions, the following several theorem tell us that the global minimum value and the local minimum value are equivalent.

Throughout this section, we let Ω be a convex set in the n - dimensional Euclidean space R^n.

Definition 3.1 Let $f: \Omega \to F^*(R)$ be a fuzzy valued mapping. Then f is called a convex fuzzy valued function defined on Ω. If for arbitrary $x, y \in \Omega, \lambda \in [0, 1]$, $f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y)$ always holds.

Especially, whenever $0 < \lambda < 1$, and above strict inequality holds, then f is called a strictly convex fuzzy valued function.

Similarly, by $f(\lambda x + (1-\lambda)y) \geqslant \lambda f(x) + (1-\lambda)f(y)$, we may define the concave fuzzy valued functions. Here, we only study the convex fuzzy valued functions.

Definition 3.2 Let the convex set $\Omega \subset R^n$, the mapping $f: \Omega \to F^*(R)$, if there exists a neighbourhood $U\ (x_0, \delta) \subset \Omega$ regarding $x_0 \in \Omega$ as a centre, σ as a radius such that $f(x) \geqslant f(x_0)$, for all $x \in U(x_0, \sigma)$. Then x_0 is called a local minimum point of a fuzzy valued function f, $f(x_0)$ is called a local minimum value of f . Where

$U(x_0, \sigma) = \{x \in \Omega : \| x - x_0 \| < \sigma\}$. and $\| \cdot \|$ is a norm in R^n .

Similarly, if there exists a point $x_0 \in \Omega$ such that $f(x) \geqslant f(x_0)$, for any $x \in \Omega$. Then x_0 is called a global minimum point of a fuzzy valued function f , $f(x_0)$ is called a global minimum value of f .

Theorem 3.1 Let f be a convex fuzzy valued function defined on the open convex set $\Omega \subset R^n$. Then the set of the points which make f get the global minimum value is a convex set.

Proof First, we prove generality conclusion.

Take arbitrary $A \in F^*(R)$, define the set $m(f, A) = \{x \in \Omega : f(x) \leqslant A\}$.

We can prove straightforward that $m(f, A)$ is a convex set.

In fact, letting $x_1, x_2 \in m(f, A)$, then $f(x_1) \leqslant A, f(x_2) \leqslant A$.

Consequently, for any $0 < \lambda < 1$. from proposition 2.1 − 2.3, we have

$$\begin{aligned} f(\lambda x_1 + (1 - \lambda) x_2) &\leqslant \lambda f(x_1) + (1 - \lambda) f(x_2) \\ &\leqslant \lambda A + (1 - \lambda) A \\ &= A \end{aligned}$$

Thus, $\lambda x_1 + (1 - \lambda) x_2 \in m(f, A)$.

i.e., $m(f, A)$ is a convex set on Ω .

Now, let x_0 be a global minimum point of f on Ω ,

writting $f(x_0) = B \in F^*(R)$.

Therefore $m(f, B) = \{x \in \Omega : f(x) \leqslant B\}$

Then $m(f, B)$ is a set of all points which make f get minimum value on Ω . By the generality conclusion above, We know that it is also a convex set.

The proof is completed.

Theorem 3.2 If f is a convex fuzzy valued function defined on the open convex set $\Omega \subset R^n$. Then an arbitrary local minimum point of f must be a global minimum point of f on Ω .

Proof Let x_0 be a local minimum point of f on Ω , i.e., there exists a σ − neighbourhood $U(x_0, \sigma)$ of x_0 such that $f(x_0) \leqslant f(x)$, $\forall x \in U(x_0, \sigma)$.

Let $x \in \Omega - U(x_0, \sigma)$, clearly, we have $x \neq x_0$.

Choose $0 < \lambda < \frac{\sigma}{\| x - x_0 \|}$

Then $\| ((1-\lambda)x_0 + \lambda x) - x_0 \| = \lambda \| x - x_0 \| < \sigma$

i.e., $(1-\lambda)x_0 + \lambda x \in U(x_0, \sigma)$.

As x_0 is a local minimum point, we can derive from that

$$f(x_0) \leqslant f((1-\lambda)x_0 + \lambda x) \leqslant (1-\lambda)f(x_0) + \lambda f(x)$$

According to the operation properties of fuzzy numbers and proposition 2.1—2.3, we know, for any $\alpha \in [0,1]$

$$(f(x_0))_\alpha \leqslant ((1-\lambda)f(x_0) + \lambda f(x))_\alpha$$
$$= (1-\lambda)(f(x_0))_\alpha + \lambda(f(x))_\alpha$$

or $[f_\alpha^-(x_0), f_\alpha^+(x_0)] \leqslant [(1-\lambda)f_\alpha^-(x_0) + \lambda f_\alpha^-(x),\ (1-\lambda)f_\alpha^+(x_0) + \lambda f_\alpha^+(x)]$

Thus, $\Rightarrow \begin{cases} f_\alpha^-(x_0) \leqslant (1-\lambda)f_\alpha^-(x_0) + \lambda f_\alpha^-(x), \\ f_\alpha^+(x_0) \leqslant (1-\lambda)f_\alpha^+(x_0) + \lambda f_\alpha^+(x). \end{cases}$

Therefore, we have

$$f_\alpha^-(x_0) \leqslant f_\alpha^-(x),$$
$$f_\alpha^+(x_0) \leqslant f_\alpha^+(x)$$

Furthermore, $(f(x_0))_\alpha \leqslant (f(x))_\alpha$ holds.

Taking advantage of the decomposition theorem with respect to fuzzy sets, we get $f(x_0) \leqslant f(x), \forall x \in \Omega$.

The proof is completed.

Theorem 3.3 Let f be a strictly convex fuzzy valued function defined on the convex set $\Omega \subset R^n$, if f has global minimum value on Ω. Then it is gotten at the unique point on Ω.

Proof Let $A \in F^*(R)$ be a global minimum value of f on Ω.

i.e., there exists $x_0 \in \Omega$ such that $A = f(x_0) \leqslant f(x), \forall x \in \Omega$.

Let $m(f = A) = \{x \in \Omega : f(x) = A\}$

Then by theorem 3.1, we know that $m(f = A)$ is a conver set.

Suppose there exists $x_1 \in \Omega$ and $x_1 \neq x_0$ such that $f(x_0) = f(x_1) = A$.

Obviously, we have $x_0, x_1 \in m(f = A)$. And so, for $0 < \lambda < 1$,

we have $\lambda x_0 + (1-\lambda)x_1 \in m(f = A)$

Hence, $f(\lambda x_0 + (1-\lambda)x_1) = A$ ………………………………………… $(*)$

As f is strictly convex fuzzy valued function, we get that

$$f(\lambda x_0 + (1-\lambda)x_1) < \lambda f(x_0) + (1-\lambda)f(x_1)$$
$$= \lambda A + (1-\lambda)A$$
$$= A$$

This contradicts (*)

Therefore, f can be only gotten unique global minimum value on Ω.

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