Interval-valued fuzzy linear spaces

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Abstract: In this paper, the concept of a interval-valued fuzzy linear space (i. e. i-v fuzzy linear space) is introduced, and its properties are shown. Then using the extension principle of i-v fuzzy sets, images and inverse-images of i-v fuzzy linear spaces under a linear transformation are studied.

Key words: i-v fuzzy sets; fields; linear space; i-v fuzzy fields, i-v fuzzy linear space.

1. Introduction

In 1975, Zadeh [7] introduced the concept of interval-valued fuzzy sets (in short, written by i-v fuzzy sets), where the values of the membership functions are intervals of real numbers instead of the real points. In [3], Gorzalczany studied i-v fuzzy sets for approximate reasoning. Meng [5] made a deep study on the fundamental theory of i-v fuzzy sets, and had shown decomposition theorem and expression theorem, etc. This makes it possible to establish some corresponding mathematics theory with respect to the classical branch. For example, Biswas's i-v fuzzy subgroups [1].

It is well-known that fuzzy linear space is an important concept in fuzzy algebras, and based on membership function valued in [0, 1], its theory has been built up by Nanda [6], Biswas [2] and Gu [4]. The present paper's purpose is to set up a theory of i-v fuzzy linear space over i-v fuzzy fields, such that it is an good extension of above mentioned works.

The rest of the paper consists of three parts. After preliminaries in section 2, the i-v fuzzy field is defined in section 3. Then in section 4, the theory of i-v fuzzy linear spaces is built up.

2. Preliminaries

Let $I=[0,1],[I]=\{[a,b]:a\leqslant b,a,b\in I\}$. For each $a\in I$, if we let $a=[a,a]$, *then* $a\in[I]$.

Let $a_t\in I,t\in T$. Then we write

$\bigvee_{t\in T} a_t=sup\ \{a_t:t\in T\}$, $\bigwedge_{t\in T} a_t=inf\{a_t:t\in T\}$

Let $[a_t,b_t]\in[I],t\in T$, we define

$\bigvee_{t\in T}[a_t,b_t]=[\bigvee_{t\in T} a_t,\bigvee_{t\in T} b_t]$, $\bigwedge_{t\in T}[a_t,b_t]=[\bigwedge_{t\in T} a_t,\bigwedge_{t\in T} b_t]$

Furtherly, for $[a_i,b_i]\in[I],i=1,2$, we define

$[a_1,b_1]=[a_2,b_2]\Leftrightarrow a_1=a_2,b_1=b_2$;

$[a_1,b_1]\leqslant[a_2,b_2]\Leftrightarrow a_1\leqslant a_2,b_1\leqslant b_2$;

$[a_1,b_1]<[a_2,b_2]\Leftrightarrow[a_1,b_1]\leqslant[a_2,b_2]$ and $[a_1,b_1]\neq[a_2,b_2]$.

Obviously, $([I],\leqslant,\vee,\wedge)$ is a complete lattice with universal bounds $[0,0]$ and $[1,1]$.

Let X be a nonempty set. A mapping $A:X\to[I]$ is said to be an i-v fuzzy set. If we let $A(x)=[A^-(x),A^+(x)],\forall\ x\in X$, then the fuzzy set $A^-:X\to I$ (resp, $A^+:X\to I$) is called the lower (resp, upper) fuzzy set of A.

We use the symbol $IF\ (X)$ to denote the set of all i-v fuzzy sets on X.

Let $A_t\in IF(X),t\in T,\forall\ x\in X$, Then we define

$\bigcup_{t\in T} A_t:(\bigcup_{t\in T} A_t)(x)=\bigvee_{t\in T} A_t(x)$;

$\bigcap_{t\in T} A_t:(\bigcap_{t\in T} A_t)(x)=\bigwedge_{t\in T} A_t(x)$;

$A_{t1}\subset A_{t2}\Leftrightarrow A_{t1}(x)\leqslant A_{t2}(x),t_1,t_2\in T$;

$A_{t1}=A_{t2}\Leftrightarrow A_{t1}(x)=A_{t2}(x),t_1,t_2\in T$.

3. i-v fuzzy fields

Definition 3. 1. Let X, Y be two sets, $f: X \rightarrow Y$ be a mapping. Then the following mappings

$$f: IF(X) \rightarrow IF(Y)$$

$$f(A)(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} A(x), & \text{if } f^{-1}(y) \neq \Phi, \forall y \in Y \\ [0,0], & \text{otherwise} \end{cases}$$

and $f^{-1}: IF(Y) \rightarrow IF(X), f^{-1}(B)(x) = B(f(x))$

where $A \in IF(X), B \in IF(Y), f^{-1}(y) = \{x \in X : f(x) = y\}$

are called an i-v fuzzy transformation and an i-v fuzzy iverse transformation, respectively.

The above method, i.e. f and f^{-1} are induced by f is called extension principle. Clearly,

$$f(A)(y) = [\bigvee_{x \in f^{-1}(y)} A^-(x), \bigvee_{x \in f^{-1}(y)} A^+(x)]$$
$$= [f(A^-)(y), f(A^+)(y)]$$
$$f^{-1}(B)(x) = [B^-(f(x)), B^+(f(x))]$$
$$= [f^{-1}(B^-)(x), f^{-1}(B^+)(x)].$$

Definition 3. 2. Let X be a field and F an i-v fuzzy set on X. If the following conditions hold:

(1) $F(x+y) \geqslant F(x) \wedge F(y), x, y \in X$;

(2) $F(-x) \geqslant F(x), x \in X$;

(3) $F(xy) \geqslant F(x) \wedge F(y), x, y \in X$;

(4) $F(x^{-1}) \geqslant F(x), x(\neq 0) \in X$.

then F is said to be an i-v fuzzy field of X, denoted by (F, X).

Proposition 3. 1. If (F, X) is a i-v fuzzy field of X, then

(1) $F(0) \geqslant F(x), x \in X$;

(2) $F(1) \geqslant F(x), x(\neq 0) \in X$;

(3) $F(0) \geqslant F(1)$

Proposition 3. 2. Let X and Y be fields, and f a homomorphism of X into Y. Suppose that (F, X) is a i-v fuzzy field of X and (G, Y) is a i-v fuzzy field of Y, then

(1) $(f(F), Y)$ is a i-v fuzzy field of Y.

(2) $(f^{-1}(G), X)$ is a i-v fuzzy field of X.

4. i-v fuzzy linear spaces

Definition 4.1. Let X be a field and (F, X) be an i-v fuzzy field of X. Let Y be an linear space over X and V be an i-v fuzzy set of Y. Suppose the following conditions hold:

(1) $V(x+y) \geqslant V(x) \wedge V(y), x, y \in Y$;

(2) $V(-x) \geqslant V(x), x \in Y$;

(3) $V(\lambda x) \geqslant F(\lambda) \wedge V(x), \lambda \in X$ and $x \in Y$;

(4) $F(1) \geqslant V(0)$

then (V, Y) is called an i-v fuzzy linear space over an (F, X).

Definition 4.2 Let (V, Y) and (W, Y) be two i-v fuzzy linear space over an i-v fuzzy field (F, X). If $W \subset V$, then (W, Y) is said to be an i-v fuzzy linear subspace of (V, Y).

Proposition 4.1. If (V, Y) is an i-v fuzzy linear space over (F, X), then

(1) $F(0) \geqslant V(0)$;

(2) $V(0) \geqslant V(x), x \in Y$;

(3) $F(1) \geqslant V(x), x \in Y$

Proposition 4.2. Let (F, X) be an i-v fuzzy field of X, and Y a linear space over X. Assume V is an i—v fuzzy set of Y. Then (V, Y) is an i-v fuzzy linear space over (F, X) iff

(1) $V(\lambda x + \mu y) \geqslant (F(\lambda) \wedge V(x)) \wedge (F(\mu) \wedge V(y)), \lambda, \mu \in X$ *and* $x, y \in Y$;

(2) $F(1) \geqslant V(x), x \in Y$.

Proposition 4.3. Let (V, Y) be an i-v fuzzy linear space over (F, X), and $W \subset V$. Then (W, Y) is an i-v fuzzy linear subspace of (V, Y) iff $W(\lambda x + \mu y) \geqslant (F(\lambda) \wedge W(x)) \wedge (F(\mu) \wedge W(y)), \lambda, \mu \in X$ *and* x, $y \in Y$.

Proposition 4.4. The intersection of a family of i-v fuzzy linear spaces is an i-v fuzzy linear space.

Proposition 4.5. The intersection of a family of i-v fuzzy linear subspaces is an i-v fuzzy linear subspace.

Proposition 4.6. Let Y and Z be linear spaces over the field X, and f

a linear transformation of Y into Z. Let (F, X) be an i-v fuzzy field of X, and (W, Z) be an i-v fuzzy linear space over (F, X). Then $(f^{-1}(W), Y)$ is an i-v fuzzy linear space over (F, X).

Proposition 4.7. Let Y and Z be linear spaces over the field X, and f a linear transformation of Y into Z. Let (F, X) be an i-v fuzzy field of X and (V, Y) be an i-v fuzzy linear space over (F, X). Then $(f(V), Z)$ is an i-v fuzzy linear space over (F, X).

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