Homomorphism between the lattices of TL-submodules *

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Abstract

In this paper, we investigate the lattice homomophisms and give some lattice homomorphic and isomorphic theorems.

Keywords: Module; TL-submodule; Lattice; Homomorphism; Isomorphism

1. Introduction

Rosenfeld [7] applied the notion of fuzzy sets to abstract algebra and introduced the notion of fuzzy groups. Since then the literature of these fuzzy algebric concepts, such as fuzzy semigroups, fuzzy subrings, fuzzy ideals, fuzzy subfelds, fuzzy submodules and so on, has been growing very rapidly. Later,Yu and Wang extended these notion and introduce the theory of TL-subalgebra,TL-subgroups,TL-ideals, and TL-submodules[1,2]. However, until now the lattice theoretic aspects of these structures have not been sufficiently explored.

In this paper, we discuss the lattice homomorphisms and prove some lattice homomorphic and isomorphic theorems.

Throughout this paper, unless otherwise stated, L always represents any given complete Brouwerian lattice with maximal element 1 and minimal element 0;T indicates a t-norm;R indicates any given ring with identity e; and M indicates any given module (i.e. left module) over R with zero θ.

2. Preliminaries

In this section, we give some definitions and results which will be used in the sequel.

Definition 2.1.[2] Let f be a mapping from X into Y, and let $\mu \in L^X, \gamma \in L^Y$ .Then L-subsets $f(\mu) \in L^Y$ and $f^{-1}(\gamma) \in L^X$, defined by

$$f(\mu)(y) = \vee_{x\in X, f(x)=y}\, \mu(x) \qquad \forall y \in Y$$

and

$$f^{-1}(\gamma)(x) = \gamma(f(x)) \qquad \forall x \in X.$$

are called , respectively, the image of μ and pre-image of γ under f.

Definition 2.2.[1] Let $\mu, \gamma \in L^Y$ and r$\in$ R. Define $\mu +_T \gamma$, -μ ,and $r\mu$ as follows:

$$(\mu +_T \gamma)(x) = \vee\left\{\mu(y)T\gamma(z) \;\middle|\; y+z=x \right\},$$

$$(-\mu)(x) = \mu(-x),$$

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$$(r\mu)(x) = \vee\{\mu(y) | ry = x\}.$$

Theorem 2.1.[1] Suppose that N is also a module over R and f is a homomorphism of M into N .let $\mu, \gamma \in L^M$, then

$$f(\mu +_T \gamma) = f(\mu) +_T f(\gamma).$$

Definition 2.3.[1] By a TL-submodule of M, we mean an L-subset μ of M which satisfies the following condition:

(M1) $\mu(\theta) = 1$.

(M2) $\mu(rx) \geq \mu(x) \quad \forall r \in R \text{ and } x \in M$.

$(M3)_T$ $\mu(x+y) \geq \mu(x) T \mu(y) \quad \forall x, y \in M$.

When $T = \wedge$,a TL-submodule of M is called an L-submodule of M.

The set of all TL-submodules and the set of all L-submodules of M are denoted by TL(M) and L(M),respectively.

Theorem 2.2.[1] The set TL(M) equipped with L-subset inclusion relation $\leq$ constitutes a complete lattice with L-subset intersection as its meet. Its maximal and minimal elements are 1_M and 1_θ ,respectively.

Theorem 2.3.[1] Let $\mu, \gamma \in TL(M)$,then $\mu +_T \gamma \in TL(M)$.

The following theorem give a method to calculate the union in TL(M).

Theorem 2.4. Let $\mu, \gamma \in TL(M)$,then

$$\mu \vee \gamma = \mu +_T \gamma \ .$$

Proof. For any $x \in M$,

$$\begin{aligned}(\mu +_T \gamma)(x) &= \vee\{\mu(y) T \gamma(z) | y + z = x\} \\ &\geq \mu(x) T \gamma(\theta) \\ &= \mu(x) \ .\end{aligned}$$

Hence $\mu \leq \mu +_T \gamma$ .We have $\gamma \leq \mu +_T \gamma$ by the symmetry.

Let $\zeta \in TL(M)$,if $\mu \leq \zeta$ and $\gamma \leq \zeta$,then

$$\begin{aligned}(\mu +_T \gamma)(x) &= \vee\{\mu(y) T \gamma(z) | y + z = x\} \\ &\leq \vee\{\zeta(y) T \zeta(z) | y + z = x\} \\ &= (\zeta +_T \zeta)(x) \\ &\leq \zeta(x) \qquad \forall x \in M.\end{aligned}$$

Therefore, $\mu +_T \gamma \leq \zeta$ .This implies $\mu +_T \gamma$ is the least TL-submodules which contains μ and γ ,that is $\mu \vee \gamma = \mu +_T \gamma$ .This complete the proof. □

3. Homomorphisms and isomorphisms

In this section, N is also a module over R.

Theorem 3.1.[1] Suppose f is a homomorphism of M into N. Then

(1) $\mu \in TL(M) \Rightarrow f(\mu) \in TL(N)$,

(2) $\gamma \in TL(N) \Rightarrow f^{-1}(\gamma) \in TL(M)$,

(3) $f(\mu +_T \gamma) = f(\mu) +_T f(\gamma)$.

Theorem 3.2. Let f is a homomorphism of M into N.A mapping $\hat{f}$ is defined by

$$\hat{f}: TL(M) \to TL(N)$$
$$\mu \to f(\mu)\ .$$

Then $\hat{f}$ is a lattice union-homomorphism.

Proof. Immediate from Theorem 2.4 Theorem 3.1. □

Let f: M → N be a epimorphism. We define the set TL(M,Kerf) as follows:

$$TL(M,Kerf)=\{\mu | \mu \in TL(M), and \mu(k) = 1, k \in Kerf\}.$$

Then we have the following isomorphic theorem.

Theorem 3.3. TL(M,Kerf) is a sublattice of TL(M), and

$$\hat{f}: TL(M,Kerf) \to TL(N)$$
$$\mu \to f(\mu)$$

is a lattice isomorphism.

In order to prove Theorem 3.3, we give a lemma first.

Lemma 3.4. Let $\mu \in TL(M,Kerf)$, then

$$\mu(x+k) = \mu(x),$$

where x∈M, k∈Kerf.

Proof. $\mu(x+k) \geq \mu(x)T\mu(k) = \mu(x)T1 = \mu(x)$.

Suppose, if possible

$$\mu(x+k) \neq \mu(x),$$

then

$$\mu(x+k) > \mu(x),$$

so

$$\begin{aligned}\mu(x) &= \mu(x+k-k)\\ &\geq \mu(x+k)T\mu(k)\\ &= \mu(x+k)T1\\ &= \mu(x+k)\\ &> \mu(x).\end{aligned}$$

This contradicts the fact $\mu(x) = \mu(x)$. Hence $\mu(x+k) = \mu(x)$. □

Proof of Theorem 3.3. For any $\mu, \gamma \in TL(M,Kerf)$ and x∈Kerf, we have

$$\begin{aligned}(\mu \vee \gamma)(x) &= (\mu +_T \gamma)(x)\\ &= \vee\{\mu(y)T\gamma(z) | y+z=x\}\\ &\geq \mu(0)T\gamma(x)\\ &= 1T1\\ &= 1.\end{aligned}$$

$$\begin{aligned}(\mu \wedge \gamma)(x) &= (\mu(x) \wedge \gamma(x))\\ &= 1T1\\ &= 1\end{aligned}$$

Therefor, $\mu \vee \gamma, \mu \wedge \gamma \in TL(M,Kerf)$. In other words, TL(M,Kerf) is a submodule of TL(M).

Now we prove that $\hat{f}$ is a lattice isomorphism.

(1) For any $\mu,\gamma \in TL(M, Kerf)$. If $\hat{f}(\mu) = \hat{f}(\gamma)$, then for any x∈M, we have

$$\begin{aligned}\hat{f}(\mu)(f(x)) &= f(\mu)(f(x)) \\ &= \vee\{\mu(x+k) | k \in Kerf\ ) \\ &= \mu(x).\end{aligned}$$

Similarly we have

$$\hat{f}(\gamma)(f(x)) = \gamma(x).$$

Hence

$$\mu(x) = \hat{f}(\mu)(f(x)) = \hat{f}(\gamma)(f(x)) = \gamma(x).$$

Therefore $\mu = \gamma$ .This prove $\hat{f}$ a injective.

(2) For any $\zeta \in TL(N)$, let $\mu = f^{-1}(\zeta)$, then $\mu \in TL(M)$ by Theorem 3.1 and for any k∈Kerf, we have

$$\mu(k) = f^{-1}(\zeta)(k) = \zeta(f(k)) = \zeta(0) = 1 .$$

Hence $\mu \in TL(M, Kerf)$ and

$$\hat{f}(\mu) = f(f^{-1}(\zeta)).$$

But for any y∈N, we have

$$\begin{aligned}f(f^{-1}(\zeta))(y) &= \vee\{f^{-1}(\zeta)(x) | f(x) = y\ \} \\ &= \vee\{\zeta(f(x)) | f(x) = y\ \} \\ &= \zeta(y).\end{aligned}$$

So, we obtain $\hat{f}(\mu) = f(f^{-1}(\zeta)) = \zeta$ and consequently $\hat{f}$ is surjective.

(3) For any $\mu,\gamma \in TL(M, Kerf)$, from Theorem 3.1 we have

$$\hat{f}(\mu \vee \gamma) = f(\mu +_T \gamma) = f(\mu) +_T f(\gamma) = \hat{f}(\mu) \vee \hat{f}(\gamma).$$

(4) For any $\mu,\gamma \in TL(M, Kerf)$ and any x∈M, we have

$$\begin{aligned}\hat{f}(\mu \wedge \gamma)(f(x)) &= f(\mu \wedge \gamma)(f(x)) \\ &= \vee\{(\mu \wedge \gamma)(x+k) | k \in Kerf\ \} \\ &= \vee\{(\mu(x+k) \wedge \gamma(x+k) | k \in Kerf\ \} \\ &= \vee\{(\mu(x) \wedge \gamma(x) | k \in Kerf\ \} \\ &= \mu(x) \wedge \gamma(x)\end{aligned}$$

and

$$\begin{aligned}(\hat{f}(\mu) \wedge \hat{f}(\gamma))(f(x)) &= (f(\mu) \wedge (f(\gamma))(f(x) \\ &= (f(\mu)(f(x))) \wedge (f(\gamma)(f(x))) \\ &= (\vee\{\mu(x+k) | k \in Kerf\ \}) \wedge (\vee\{\gamma(x+k) | k \in Kerf\ \}) \\ &= \mu(x) \wedge \gamma(x)\ .\end{aligned}$$

Therefore

$$\hat{f}(\mu \wedge \gamma) = \hat{f}(\mu) \wedge \hat{f}(\gamma).$$

This complete the proof. □

Corollary 3.5. Let K be a submodule of M, then

$$TL(M,N) \cong TL(M/N),$$

where $TL(M,N)=\{\mu | \mu \in TL(M), and \mu(n) = 1, n \in N\}$

Remark. From the discussion above, we can find that if homomorphism f : M → N is not surjective, then $\hat{f}$ is also a lattice monomorphism .

Let f is same as theorem 3.3 and $\mu \in TL(M)$. We can define the set $TL(\mu, Kerf)$ as follows:

$$TL(\mu, Kerf) = \{\mu | \mu \in TL(\mu) \text{ and } \mu(k) = 1, k \in Kerf\}.$$

Where $TL(\mu) = \{\eta | \eta \in TL(M) \text{ and } \eta \le \mu\}$.Then we have the following isomorphic theorem:

Theorem 3.6. $TL(\mu, Kerf)$ is a sublattice of $TL(M, Kerf)$,and

$$\hat{f}: TL(\mu, Kerf) \to TL(f(\mu))$$
$$\gamma \to f(\gamma)$$

is a lattice isomorphism.

Proof. Similar to the proof of theorem 3.3. □

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