New Operations in Algebra of Fuzzy Sets

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Abstract: In this paper a new operations in algebra of fuzzy sets are given. Algebra of fuzzy sets is defines operations multiplication, addition, inverse elements, etc. All of these operations are obtained for mathematical modeling fuzzy processes [4-6]. The structure of mathematical operations allows solving algebraic equations. Operations are defined by using special conception of fuzzy sets. Fuzzy sets provide a mathematical notation for representing real world concepts with are essentially vague.

Keywords: Algebra of fuzzy sets, fuzzy sets group, fuzzy operations.

1. Introduction.

This paper is directed to the development of a special algebra of fuzzy sets. Fuzzy sets suggested in papers [1-2] offer a possibility to formally describe linguistic expressions like tall, fast, medium, etc., and to operate on them. This paper has formal approach. It means that fuzzy set is the graph of function, which specific conditions are fulfilled. Each fuzzy set is associated with the real number. Algebra of fuzzy sets underlies thus real numbers algebra. We want to add fuzzy sets in the same way as we add real numbers. Other operations need to have the same properties. It will allow the use of the traditional methods of the theory of control for the analysis of fuzzy systems. The application of algebra enables easier and more convenient description fuzzy processes [4-6].

It is necessary to provide a fuzzy set with relationships with real numbers, as fuzzy sets are much more complicated objects than real numbers. However, simple direct duplication of group operations of the real numbers to the fuzzy sets may not be adequate[9]. Nevertheless, we require that the algebra of defuzzificated values should remind the algebra of real numbers.

It is convenient to define group operations[7-8] on the axis G of membership degrees and on the axis X of the domain of definition separately. This refers for each argument $x \in X$. We compute a value on the axis G for each argument $x \in X$, and then introduce the final operation on the axis X. Operations with each element of the set $H=X \times G$ makes an algebra for H. Thus H is a system of sets with the introduced algebraic operations. The present below theory enables to define the concepts of the metric, the vector space[7-8], etc.

The paper is organized as follows. In the next section the basic definitions are given. The problem is also formulated in section 2. Third section is aimed to the construction of an algebra on a segment. Section 4, we suggest a definition of a ring on the axis X. In section 5 and 6 the theoretical concepts are exemplified with a few simple examples of the vector space. The last section, 6, summarizes our findings and described types of membership functions.

2. Definitions.

For the sake of completeness we give here the basic definitions [1-2].

Definition 1. Fuzzy set (FS) A of X is a function defined on the universe X which represents a mapping by means of membership function μ

$$\mu_A: F(R) \to G \tag{2.1}$$

Where $F(X)=\{\mu / \mu: X \rightarrow [0;1]\}$ is set of all fuzzy subsets of the universe, and G is a range of values of fuzzy set.

As a set of membership degrees, the unit segment $G=[0;1]$ is mostly chosen. Thus, the segment $[0;1]$ is adopted as a definition of a set G in what follows. The number $\mu(x) \in [0;1]$ may be considered as the membership degree of x to the fuzzy set μ. As a definition of the universal set X (2.1) we take a set of elements of the real axis $X=R\equiv(-\infty; +\infty)$. With the help of the same using the chosen defuzzification method any fuzzy set can be associated with unique real number from the set R (the opposite statement is not true).

The fuzzy set is defined on the two-dimensional space $H=\{X;G\}$. The membership function μ is defined on a set $F(R)$ and takes values from a set G. It is convenient to represent μ as a graph where the axis of abscissas is the set X and the axis of ordinates is the set G.

This study is aimed to construct an algebra of numbers defined above. First of all we need to define operations on a set of fuzzy sets. Such operations are operations of abstract multiplication and abstract addition. As long as we have a set with two operations defined on this set, we can introduce a ring. Then, we can deduce the algebra of fuzzy sets defining a vector space on the field of real numbers.

At first, the definition to a ring of fuzzy sets reads[7-8].

Definition 2. A ring H is defined as a class of objects with the operations of abstract addition and abstract multiplication $(\oplus, \otimes)$ such that

1. H is commutative group with respect to addition . This means, that the properties of commutativity, associativity, existence of zero, and inverse of the element are fulfilled

$$a\oplus b=b\oplus a \ ; \ a\oplus(b\oplus c)=(a\oplus b) \oplus c \ ; \ a\oplus 0=a \ ; \ a\oplus(-a)=a-a=0 \qquad (2.2)$$

2. H has a closure in relation to multiplication (the second group operation is introduced) and the associativity on multiplication and distributive laws are fulfilled

$$a\otimes(b\otimes c)=(a\otimes b) \otimes c \ ; \ a\otimes(b\oplus c)=a\otimes b\oplus a\otimes c \ ; \ (b\oplus c) \otimes a=b\otimes a\oplus c\otimes a \qquad (2.3)$$

Turn now to the principal definitions of our work [7].

Definition 3. Algebra is defined as a ring H, which is a vector space on the field F, if for $\forall\lambda \in F$ and $\forall x,y \in H$ the following is fulfilled

$$\lambda(xy)=(\lambda x)y=x(\lambda y) \qquad (2.4)$$

Definition 4. A vector space is defined as a commutative group H in a commutative skew field F, if $\forall\alpha,\beta\in F$ and $\forall u,v\in H$ the following is fulfilled

1. $\exists\ \alpha u \in H$ (2.5)
2. $(\alpha+\beta)u=\alpha u+\beta u;\ \alpha(u+v)=\alpha u+\alpha v;\ (\alpha\beta)u=\alpha(\beta u)$ (2.6)
3. $\forall 1 \in F$ and $\forall u \in H$ is fulfilled $1u=u$ (2.7)

For the sake of simplicity we will build the algebra H (definition 3) on the skew field of the real numbers and not on the abstract skew field as usually. Further on, we prove that the addition group which we study in this paper is Abelian.

The basic difference between fuzzy sets and common numbers is the two-dimensional object[1]. The fuzzy set has a range of definition G and a range of values X, in which the specific conditions should be fulfilled. To introduce the algebraic structures on these sets it is necessary to define algebra on the axis of the membership degrees G and on the axis of the universe X separately.

The algebra for two-dimensional objects is defined as follows. We compute the result of the operation on the axis of the universe X. Simultaneously, in the range of values for membership degrees of the elements the result is computed.

Definition 5. The result of addition of two elements $\{x_1;\mu_1(x_1)\}$ and $\{x_2;\mu_2(x_2)\}$ on the graphs of fuzzy sets μ_1 and μ_2 defined in (1.1) there is a element on the fuzzy set determined as follows

$$\{x_1;\mu_1(x_1)\}\oplus_1\{x_2;\mu_2(x_2)\}=\{(x_1\oplus_2x_2);(\mu_1(x_1)\oplus_3\mu_2(x_2))\} \tag{2.8}$$

Where $\oplus_1$ is addition of elements on the graph of fuzzy sets μ_1 and μ_2, $\oplus_2$ is addition assigned on the axis of the universe X, $\oplus_3$ is addition defined on the axis of the membership degree G of fuzzy set. Working on various elements on the graphs of fuzzy sets we obtain a sum of fuzzy sets. We need to isolate those elements that concern the adding operation for uniqueness of the produced operation.

It is possible to define other elementary operations of algebra of fuzzy sets. The logic of human reasoning suggests that the addition $\oplus_2$ should remind the addition of real numbers in many aspects. The more detailed analysis given in the further algebra enables us to select an ideal.

Definition 6. An ideal I is defined as subsets of a ring, provided that the following conditions are fulfilled

1. I is a subgroup of addition.
2. I contains all products $a\otimes b$, where a is any element from I, and b is any element of the ring.

It is now necessary to define an algebra on the axis G of fuzzy set. One group operation $\oplus_3$ is discussed in the definition 5 as an addition of two fuzzy sets. The following section is dedicated to the definition of this operation and to the multiplication operation.

3. Algebra of membership degree of a fuzzy set.

Before constructing of a ring (definition 2) of fuzzy sets it is necessary to define operations which for any numbers x_1 and x_2 (membership degrees of two fuzzy sets) have a correspondent number x from a unit segment $x\in G=[0;1]$, provided that x_1 and $x_2\in G=[0;1]$. Each operation should gives rise to a group while two operations dive rise to a ring.

The algebra operations on the axis G of membership functions should keep the resulting elements within the segment $[0;1]$, therefore these operations should be contracting: the sum is always less than the result of the common addition operation. Introduce now the addition operation for a class of objects $G=[0;1]$.

Definition 7. The contracting sum $\oplus_3$ puts in a correspondence with the two elements x_1 and $x_2\in[0;1]$ the element $x=x_1\oplus_3x_2$ by the rule

$$x=x_1\oplus_3x_2=W(W^{-1}(x_1)+W^{-1}(x_2)) \tag{3.1}$$

Where W is a monotonically increasing function with its values within the interval $(0;1)$. Fig.1 exemplifies such a function

$$W(t)=1/(1+e^{-t}) \tag{3.2}$$

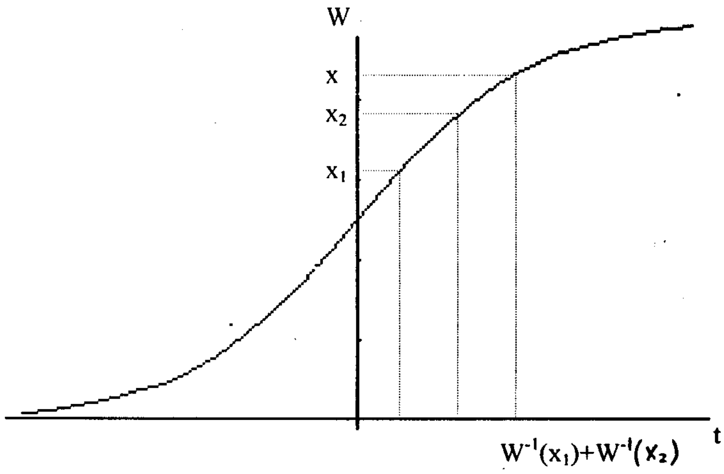

Fig.1 Contracting addition.

The segment *[0;1]* (axis of value of function *W*) is a class of objects *G*. Figure 1 explains the contracting addition. On Fig.1 auxiliary an axis *t* is shown and the area *G* is the area of possible values of the function *W*. Let's consider two elements x_1 and x_2 on an axis *G*. They are projected on an additional axis *t* by the reverse mapping $W^{1}(x)$. Images of these elements $W^{1}(x_1)$ and $W^{1}(x_2)$ can be combined as the real number on axis. The image of a sum of two elements $W^{1}(x_1)+W^{1}(x_2)$ is obtained. With the help of functions *W* (in according with (3.2)) we project the image of a sum back onto the interval *G=[0;1]*. The addition rule introduced demonstrates a number advantages: for any two numbers x_1 and x_2 always puts in a correspondence a number greater than each of the two given; also many properties of addition of real numbers (e.g. commutativity) remain unchanged.

In the same way one can define operation of multiplication on a ring *G*.

Definition 8. The contracting multiplication $\otimes_3$ puts in a correspondence with two elements x_1 and $x_2 \in [0;1]$ the element $x=x_1\otimes_3 x_3$ by the rule

$$x=x_1\otimes_3 x_2=W(W^{1}(x_1)W^{1}(x_2)) \qquad (3.3)$$

Where *W* is a monotone increasing function defined on the real axis. For simplicity we take the same function as for the contracting sum used above.

Again, in our definition of contracting multiplication many properties of operations with the real numbers are preserved. The axis *G* defines a membership degree of fuzzy set to a particular class of objects. This membership degree is number from a segment *[0;1]*. Therefore, in the definition of a contracting sum and contracting multiplication are should take into account that the sum of two membership degrees x_1 and x_2 should belong to the unit segment.

Ease to prove, that the operation $\oplus_3$ defines commutative infinite group on *G=[0;1]*, i.e. that the property (2.2)-(2.3) is fulfilled

The group *G* also contains zero: $E=W(0)$ for $\forall a \in G$. Indeed,

$E\oplus_3 a=W(W^{1}(E)+W^{1}(a))=W(W^{1}(W(0))+W^{1}(a))=W(W^{1}(a)=a)$

Left zero coincides with right zero. It means that for $\forall a$ from the equation $a\oplus_3 E_1=a$ follows: $E_1=E$. Thus we have a sole zero.

For each element *b* from *G*, *G* has an inverse element b^{-1} defined by the rule: $b^{-1}=-b=W(-W^{1}(b))$ so that

$$b^{-1}\oplus_3 b=W(W^{1}(W(-W^{1}(b)))+W^{1}(b))=E=b\oplus_3 b^{-1} \qquad (3.4)$$

The addition group is an Abelian group. The second operation determines group *G* on a ring. The inverse element on multiplication is defined similarly to that of the real numbers

$$a^{-1}=1/a=W((W^{1}(a))^{-1}) \qquad (3.5)$$

Then a unit element on a ring of multiplication will be a number $N=W(1)$. To show that $N=W(1)$ is indeed a unit we write

$N\otimes_3 a=W(W^{1}(W((W^{1}(a))^{-1})W^{1}(a))=W(W^{1}(a))=a$

also: $a^{-1}\otimes_3 a=N$. It is also worth to note that

$a\otimes_3 E=W(W^{1}(a)W^{1}(W(0)))=W(0)=E$.

Let us analyze the addition and multiplication operation

$a\otimes_3(b\oplus_3 c)=a\otimes_3 W(W^{1}(b)+W^{1}(c))=W(W^{1}(a)W^{1}(W(W^{1}(b)+W^{1}(c))))=$

$=W(W^{1}(a)W^{1}(b)+W^{1}(a)W^{1}(c))=W(W^{1}(a\otimes_3 b)+W^{1}(a\otimes_3 c)=a\otimes_3 b\oplus_3 a\otimes_3 c$

I.e. the distributive law are satisfied. The other law (2.3) may be similarly checked: $(b\oplus_3 c)\otimes_3 a=b\otimes_3 a\oplus_3 c\otimes_3 a$.

In a ring G with operations of contracting addition and contracting multiplication one can define an ideal (definition 6). Let us consider the unit element of a ring G. We define $W^{1}(1)=+\infty$, which appears as an asymptotic limit

$$\lim_{x\to 1} W^{-1}(x)=+\infty \qquad (3.6)$$

The unit element has an important property: $\forall x\in\{G\backslash\{1;0\}\}$ and for the unit element the following relations are fulfilled: $x\oplus_3 1=1$ and $x\otimes_3 1=1$. We prove here the first relation

$x\oplus_3 1=W(W^{1}(x)+W^{1}(1))=W(+\infty)=1$

the second statement is similarly proved

$x\otimes_3 1=W(W^{1}(x)W^{1}(1))=W(+\infty)=1$, i.e. $W(+\infty)=1 \Leftarrow W^{1}(1)=+\infty$

The element 1 and element 0 as elements of the group G form a subgroup of addition. We can define operations for the element 0. Starting from the definition $W^{1}(0)=-\infty$, which is an asymptotic

$$\lim_{x\to 0} W^{-1}(x)=-\infty \qquad (3.7)$$

The inverse element of 0 (on addition) is the element 1 in the ring. The equality $1\oplus_3 0=W(0)$ (written in a group form $a\oplus_3(-a)=0$) is valid.

$1\oplus_3 0=W(W^{1}(1)+W^{1}(0))=W(0)=E$

here E is zero element of our ring. To confirm that the set of elements $I=\{0,1\}$ forms an ideal we check that for the set I, $a\otimes_3 b\in I$ for $a\in I$, and $b\in G$

$$1\otimes_3 b=W(W^{1}(1)W^{1}(b)=W(+\infty)=1\in I \text{ (at } b>W(0)) \qquad (3.8)$$

$$0\otimes_3 b=W(W^{1}(0)W^{1}(b)=W(-\infty)=0\in I \text{ (at } b>W(0)) \qquad (3.9)$$

In the case of $b<W(0)$ may be analyzed analogously.

Let's summarize. In construction algebra of fuzzy sets we introduce the addition and multiplication operations of membership degrees. For the complete definition of addition $\oplus_1$ (based on the formula 2.8) we need to define addition $\oplus_2$, since addition $\oplus_3$ is already defined.

We have obtained an Abelian or commutative ring with the unity, on which there is an ideal I. The choice of function W defines uniquely the operations. If the membership degrees G are not defined in the segment $[0;1]$, but within any other closed interval (e.g. $[-1;1]$), the function W constructed on the segment $[-1;1]$. It is easy to prove that the function W defines the same ring. One can choose a function W, symmetric with respect to $W(0)$ then the contracting operations will be easier to be understood or to be selected. If the area of a membership degree is the distributive lattice, it is possible to define a step-function W, where all the operations of addition and multiplication do exist.

It is worth to note that the resulting sum and the result of the product contracting operations belong to $[0;1]$.

4. Algebra of the universe.

This section is directed to discussing the properties, which should satisfy a particular addition $\oplus_2$, so that it will be easy to select the remaining operations in a ring. Two basic methods of addition will be considered. One can define addition and multiplication in different ways depending on the particular requirement for the algebra of fuzzy sets.

Definition 9. The operator $df(\mu)$ (defuzzification abbreviation) produces defuzzification in a fuzzy set μ, and the result of this operation is a defuzzificated value $x=df(\mu(x))\in X$.

Example. The simplest way to define an algebra is to use the operations of contracting addition and multiplication on the axes of membership degrees, setting values from a range of definitions of membership functions. If we consider two sets a and b shown in Fig.2, the outcome addition $c=a+b$ will be a function shown in Fig.3.

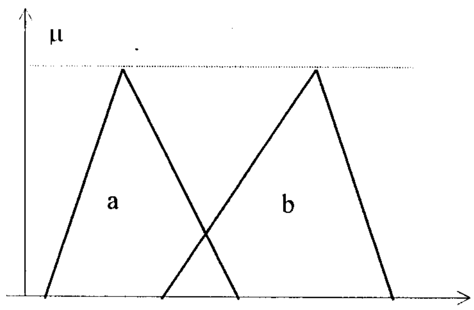

Fig.2 Fuzzy sets for addition.

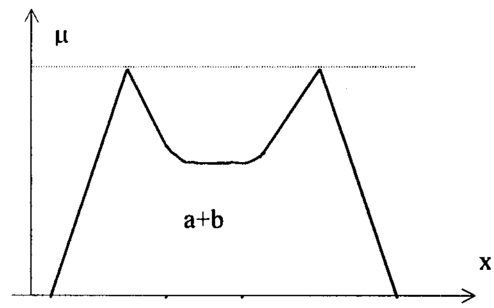

Fig.3 Addition of fuzzy sets.

This addition is produced for each element $x\in X$. It means, that in the formula (2.8) we do not distinguish a element x_1 from a element x_2, and we add only memberships of various fuzzy sets μ_1 and μ_1 for the element x.

Definition 10. The addition without a displacement $\oplus_1$. It is an addition of fuzzy sets in which the modified equality (1.8) is fulfilled

$$\{x;f_1(x)\}\oplus_1\{x;f_2(x)\}=\{x;(f_1(x)\oplus_3 f_2(x))\} \qquad (4.1)$$

So the defined addition of fuzzy sets has a number of advantages. First, knowing a resulting sum from $\mu=\mu_1\oplus_1\mu_2$ and one from the two-place operation fuzzy sets (for example μ_1), we can have (by subtraction) a fuzzy set μ_2.

There is one draw back in definition 13. If we produce a defuzzification of the sets μ_1, μ_2 and $\mu=\mu_1\oplus_1\mu_2$, it is quite possible, that the defuzzificated value μ lays between μ_1 and μ_1. Then the inequality holds

$$df(\mu_1)\leq df(\mu)\leq df(\mu_2) \qquad (4.2)$$

It does not correspond to our expectation of addition properties of two numbers (for example: $1+3=2$). There are two ways to overcome the problem. One can select method of defuzzification, the other way is to change a circuit of addition of fuzzy sets. We will change the addition of fuzzy sets, but at expense of the contraction of functions class used.

We now try to complete this task, bearing in mind that the algebra will coincide with the common numerical algebra used for the defuzzification of fuzzy sets. To complete this task we consider the membership functions applied to symmetric functions, where each function can be defined as follows.

Definition 11. The membership functions applied to symmetric functions have the following property: $\exists a \in R$ such that $\forall x \in X\ f(a-x)=f(a+x) \in [0;1]$

If we have two symmetric fuzzy sets a and b after defuzzification they will be presented as number a_d and b_d. We can now present each function a and b as a system of functions

$$a = \begin{cases} a^+(x) = a(x),\ for\ \ x > a_d \\ a^-(x) = a(x),\ for\ \ x < a_d \end{cases} \tag{4.3}$$

Examples of symmetric functions may be functions isosceles triangles. Let's define the addition operation.

Definition 12. An addition with displacement $\oplus_l$. We have two fuzzy sets a and b. One can define the function $c=a+b$. The center of defuzzification is a element $df(c)=df(a)+df(b)$. In this element the following system should be satisfied

$$\begin{cases} c(df(c)-x) = a(df(a)-x) \oplus_3 b(df(b)-x),\ for\ \ x < df(c) \\ c(df(c)+x) = a(df(a)+x) \oplus_3 b(df(b)+x),\ for\ \ x > df(c) \end{cases} \tag{4.4}$$

It is the definition of the addition with displacement. According to this formula the function which is obtained is a contracting addition of functions a and b, but in a different area. The characteristic example is considered in Fig.4.

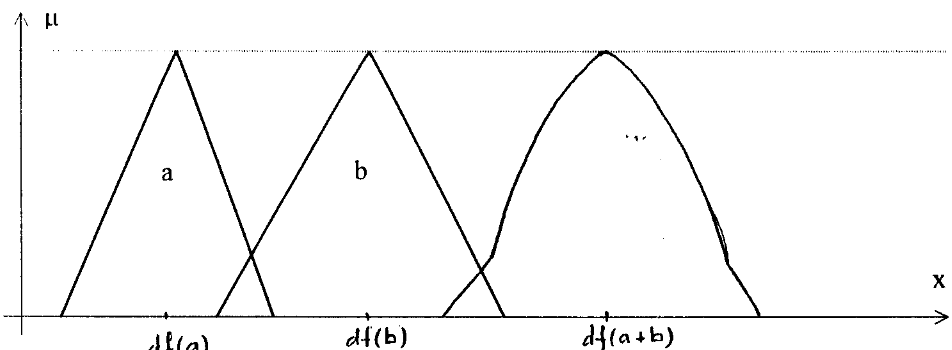

Fig.4 Addition with displacement.

We now need two basic requirements. The addition in a universal set in exactitude corresponds to common representation of addition. And the membership degrees do not go out from the frameworks of a unit segment. There is a possibility of a solution of elementary equations. Now, we are to solve the equation $a+x=b$. We write $-a+a+x=-a+b \Rightarrow x=b-a$ [8]. If we have two fuzzy sets a and b we can find fuzzy set $a=c\oplus_l(-b)$ (3.5) in a defined class of functions. This is a new possibility, which was absent in classical approach.

It is necessary to notice that the resulting fuzzy set $a\oplus_l b$ has the largest subset in the universe X. We can have the maximum width among a and b.

It is possible to consider the other types of algebra on the axis X. For example, one can consider all fuzzy sets in a unit square, where both axis of abscissas and axis of ordinates have values included in a segment $[0;1]$. On this segment the contracting operations of addition and multiplication will act following the same directions. That means they will develop both the values of the function and of the membership degree and so $\oplus_2=\oplus_3$. Such algebra does not correspond common minding, though it is a way to define the univalent algebra in a unit quadrate.

5. Construction of a vector space.

We turn now to the discussion of a vector space. In the previous two sections the place for the construction of a ring on fuzzy sets is assigned. Consider again the definition 5. In the correspondence given in it, (except for the case of a ring) it is still necessary to construct a vector space on a field *F*. We call the field of real numbers as a field *F*. We try to construct such a structure on fuzzy sets, so that it will be similar to the elementary operations applied to real numbers. It is necessary to draw an analogy between fuzzy sets and real numbers.

We need to construct a vector space in correspondence with definition 4. A basic problem here is to find a way to produce a multiplication (2.4) λu where $\lambda \in R=F$ and $u \in H$. First, the defuzzificated values of fuzzy sets must be similar to the values of real numbers. It means, that with $v=\lambda u$, the equality holds

$$df(v)=\lambda df(u) \tag{5.1}$$

There is another condition that we have to impose on the form of fuzzy set, in a multiplication λu is the necessity for every $x \in X$, the memberships $v=\lambda u$ to vary proportionally to $|\lambda|$. To find a solution we shall consider the realization of the following elementary operation

$$-1u=-u \tag{5.2}$$

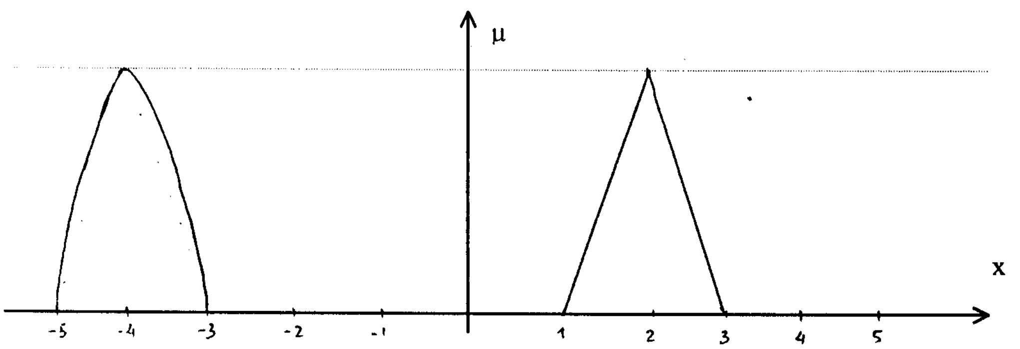

Fig.5 multiplication of fuzzy set *u* on a real number $\lambda=-2$

If we multiply a fuzzy set u on a scalar $\lambda=-1$, then the fuzzy set is mapped symmetrically to zero in the area of the negative real numbers. Note that the form or the functional association of fuzzy set does not vary. A form modification should happen in the case when $\lambda\neq|1|$. For multiplication of fuzzy sets with scalars we shall put aside values $\lambda \in R$ on an additional axis *t* of function *W*. Then we shall receive a modification in a membership degree of fuzzy set (look the definition of a contracting sum).

The inverse map in the area of membership degrees is possible when the element λx (where $\lambda \in R$) is calculated so that the value $x \in G$ is calculated by multiplication of an image $W^{-1}(x)$ and the number λ is from on an additional axis *t*.

<u>Definition 13</u>. The scalar λ multiplied on a fuzzy set *u* represents a fuzzy set *v* and it is defined by the equality

$$v(x)\equiv\lambda\otimes_1 u(x)=W(W^{-1}(u(x-df(u(x))(\lambda-1))\lambda)) \tag{5.3}$$

The formula (5.3) is explained above and the example of multiplication of fuzzy set *u* on a real number $\lambda=-2$ is submitted on Figure 5. The formulas of definition 4 are checked with the help of logic reasoning.

6. Summery discussion.

This section is directed to many aspects of the addition operation. In the definition 12 we were restricted to consider only symmetric functions. It is even easier to use a class of functions which are the normal function, as there are no complexities with the hit of the values in a scope of an ideal *I*.

Before a construction in algebra it is always necessary to set a class of functions. For any defined ring it is possible to select such tailored functions which do not belong to the algebra of a considered ring.

For creation of a ring on fuzzy sets it is better not to use extreme operations (*min* and *max*). It is impossible to restore the number even through the use of the inverse element [9]. For example, in the expression $a+b=c$, only *c* contains information about *a* so that the expression $b=c-a$ will not give us any information about the original value of *b*.

As it was clamed above, in the algebra defined in a section 2 always an ideal exists. It means, that it is possible to select such functions obtained by the range of values *G* which will be swallowed by the other functions. Thus, after the addition of a function *x* with the function of ideal, it is impossible by a subtraction to restore the original function *x*. If we use only symmetric and only subnormal function we shall receive algebra described above.

7. Conclusions.

Finally a few remarks should be made. The main idea of this paper is to construct a system of fuzzy sets by the elementary group of operations of addition and multiplication. Also we want to take into account the important requirements. Some performances of algebra should correspond to common representations.

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