Role of clans in the proximities of intuitionistic fuzzy sets

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Abstract : Definitions of filters, grills, clans and proximities are given in intuitionistic fuzzy setting. It is proved that proximities of intuitionistic fuzzy sets is a clan generated structure.

Keywards : Intuitionistic fuzzy sets, filters, grills, clans and proximities of intuitionistic fuzzy sets.

0. Introduction

In [1] K.Atanassov and S.Stoeva defined intuitionistic fuzzy sets. Later on several authors worked on intuitionistic fuzzy sets. Among others mention may be made of Atanassov [2], [3], [4], Burillo and Bustince [5], [6], D.Çoker [8], [9] and Samanta et. el. [10], [11]. Atanassov and Bustince mainly worked on several operators and algebraic properties of intuitionistic fuzzy sets ; where as D.Çoker, Samanta et. el. worked on topological structures of intuitionistic fuzzy sets.
In [7] Chattopadhyay, Samanta and Mukherjee fuzzified an important result of classical proximity by proving that proximities of fuzzy sets are clan generated structer. In this paper we define a pre-proximity and a proximity of intuitionistic fuzzy sets and prove that proximities of intuitionistic fuzzy sets are clan generated structures.

1. Preliminaries and Notations

Definition 1.1 [1] Let X be a nonempty fixed set. An intuitionistic fuzzy set (IFS in short) A

is an object having the form

$$A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$$

where the functions $\mu_A, \nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of the element $x \in X$ to the set A respectively and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, for each $x \in X$.

Example 1.2 [1] Every fuzzy set A on a nonempty set X is obviously an IFS having the form

$$A = \{< x, \mu_A(x), 1 - \mu_A(x) >: x \in X\}.$$

Notation 1.3 IFSs are denoted by A, B, C, D etc with (or without) suffix. Set of all IFSs on X are denoted by $I(X)$.

Definition 1.4 [1] Let $A, B \in I(X)$. Then
(a) $A \subset B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x), \forall\, x \in X$,
(b) $A = B$ iff $A \subset B$ and $B \subset A$,
(c) $A^c = \{< x, \nu_A(x), \mu_A(x) >: x \in X\}$,
(d) $A \bigcap B = \{< x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x) >: x \in X\}$,
(e) $A \bigcup B = \{< x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x) >: x \in X\}$.

Definition 1.5 [8] $\tilde{0} = \{< x, 0, 1 >: x \in X\}$ and $\tilde{1} = \{< x, 1, 0 >: x \in X\}$.

Corollary 1.6 [8] Let $A, B \in I(X)$. Then
(a) $(A \bigcup B)^c = A^c \bigcap B^c$,
(b) $(A \bigcap B)^c = A^c \bigcup B^c$,
(c) $(\tilde{1})^c = \tilde{0}$,
(d) $(\tilde{0})^c = \tilde{1}$.

Definition 1.7 For $x \in X$, $p \in (0, 1]$, $q \in [0, 1)$ with $p + q \leq 1$, an IFS A s.t.
$\mu_A(x) = p, \ \nu_A(x) = q$
and
$\mu_A(y) = 0, \ \nu_A(y) = 1, \ \forall\, y (\neq x) \in X$
is called an intuitionistic fuzzy point (in short IFP) on X. This is denoted by $(p, q)_x$.

Notation 1.8 We denote J as an indexing set.

2. Filter, Grill, Prime filter of intuitionistic fuzzy sets

Definition 2.1 A stack S of IFSs on X is a subset of $I(X)$ such that $A \supset B \in S \Rightarrow A \in S$.

Definition 2.2 A filter F of IFSs on X is a subset of $I(X)$ satisfying the following :
$F \neq \phi$
$A \supset B \in F \Rightarrow A \in F$
$A, B \in F \Rightarrow A \bigcap B \in F$.

A filter F of IFSs is called proper if $\tilde{0} \notin F$.

Definition 2.3 A Grill G of IFSs on X is a subset of $I(X)$ satisfying the following :
$\tilde{0} \notin G$
$A \supset B \in G \Rightarrow A \in G$
$A \bigcup B \in G \Rightarrow A \in G$ or $B \in G$.

A grill G of IFSs is called proper if $G \neq \phi$.

Definition 2.4 A stack V of IFSs on X is a prime filter of IFSs on X if it is a filter of IFSs on X and as well as a grill of IFSs on X.

A maximal proper filter U of IFSs is called an ultrafilter of IFSs.

$\phi(X)$ = Set of all filters of IFSs on X.
$\Gamma(X)$ = Set of all grills of IFSs on X.
$\omega(X)$ = Set of all prime filters of IFSs on X.

Example 2.5 Let $A \in I(X)$. Define $F \subset I(X)$ by

$$F = \{B \in I(X) : B \supset A\}.$$

Now clearly $F \neq \phi$. Let $C \supset B \in F$. Then $C \supset B \supset A$ and hence $C \in F$. Again let $B, C \in F$ and so $\mu_B(x) \geq \mu_A(x)$, $\nu_B(x) \leq \nu_A(x)$ and $\mu_C(x) \geq \mu_A(x)$, $\nu_C(x) \leq \nu_A(x)$. It follows that $\mu_B(x) \wedge \mu_C(x) \geq \mu_A(x)$ and $\nu_B(x) \vee \nu_C(x) \leq \nu_A(x)$. Thus $B \bigcap C \supset A$ and therefore $B \bigcap C \in F$. Consequently F is a filter of IFSs.

Example 2.6 Let $p > 0$ and X be a nonempty set. Then

$$V_{p_x} = \{A \in I(X) : \mu_A(x) \geq p\}$$

is a prime filter of IFSs on X.

Remark 2.7 It is to be noted that for IFP $(p,q)_x$, the collection

$$V_{(p,q)_x} = \{A \in I(X) : (p,q)_x \tilde{\in} A\}$$

is a filter but in general not a prime filter. In fact, it may not be a grill. To justify this take an ordinary set $X \neq \phi$. Let p = 0.2, q = 0.3 and a fixed $x \in X$. Let $A, B \in I(X)$ with $\mu_A(x) = 0.25$, $\nu_A(x) = 0.35$, $\mu_B(x) = 0.15$, $\nu_B(x) = 0.25$. Then $A \bigcup B \in V_{(p,q)_x}$ but $A \notin V_{(p,q)_x}$ and $B \notin V_{(p,q)_x}$.

Theorem 2.8 *Let F^1, $F^2 \in \phi(X)$ and G^1, $G^2 \in \Gamma(X)$. Then*
(1) $F^1 \bigcap F^2 \subset G^1 \Rightarrow F^1 \subset G^1$ or $F^2 \subset G^1$
(2) $F^1 \subset G^1 \bigcup G^2 \Rightarrow F^1 \subset G^1$ or $F^1 \subset G^2$.

Theorem 2.9 *Intersection of filters of IFSs is a filter of IFSs.*

Theorem 2.10 *Union of grills of IFSs is a grill of IFSs.*

Definition 2.11 For each stack S of IFSs, define $dS = \{A : A^c \notin S\}$.

Theorem 2.12 *If S (with or without suffixes) is a stack of IFSs, F is a filter of IFSs and G is a grill of IFSs on X, then followings hold :*
(1) $dS^1 \subset dS^2$ if $S^1 \supset S^2$,
(2) $d(dS) = S$,
(3) $d(\bigcup S^i) = \bigcap dS^i$,
(4) $d(\bigcap S^i) = \bigcup dS^i$,
(5) dF is a grill of IFSs,
(6) dG is a filter of IFSs.

Theorem 2.13 *If F is a filter of IFSs and G is a grill of IFSs such that $F \subset G$ then there exists a prime filter V of IFSs such that $F \subset V \subset G$.*

Corollary 2.14 Let $G \subset I(X)$. Then G is a grill of IFSs on X iff it is a union of prime filter of IFSs on X.

3. Proximities of IFSs

Definition 3.1 A binary relation Δ on $I(X)$ is said to be a basic preproximity of IFSs on X if it satisfies the following conditions :
(1) $\tilde{0} \notin \Delta(A)$, $\forall\ A \in I(X)$,
(2) $\Delta = \Delta^{-1}$,

(3) $A \bigcup B \in \Delta(C) \Leftrightarrow A \in \Delta(C)$ or $B \in \Delta(C)$ where $\Delta(A) = \{B \in I(X) : (A, B) \in \Delta\}$.

A binary relation π on $I(X)$ is said to be a basic proximity of IFSs on X if it is a preproximity of IFSs and X satisfies the condition

$$A \bigcap B \neq \tilde{0} \Rightarrow (A, B) \in \pi.$$

When $\Delta(\pi)$ is a preproximity (proximity) of IFSs on X then X is called the reference set of $\Delta(\pi)$ and is denoted by $X(\Delta)(X(\pi))$.
Set of all basic preproximities (proximities) of IFSs on X is denoted by $m(X)(M(X))$.
In the sequel, we shall, in general, drop the prefix 'basic' and just talk of preproximities of IFSs and proximities of IFSs. The pair $(X, \Delta)((X, \pi))$ is called a preproximity space of IFSs (proximity space of IFSs) whenever $\Delta \in m(X)(\pi \in M(X))$.

Example 3.2 Let $T = \{(A, B) \in I(X) \times I(X) : A \bigcap B \neq \tilde{0}\}$. Then $(A, \tilde{0}) \notin T$ and $(A, B) \in T \Rightarrow (B, A) \in T$. Now

$$\begin{aligned}
(A, B \bigcup C) \in T &\Leftrightarrow A \bigcap (B \bigcup C) \neq \tilde{0} \\
&\Leftrightarrow (A \bigcap B) \bigcup (A \bigcap C) \neq \tilde{0} \\
&\Leftrightarrow A \bigcap B \neq \tilde{0} \text{ or } (A \bigcap C) \neq \tilde{0} \\
&\Leftrightarrow (A, B) \in T \text{ or } (A, C) \in T.
\end{aligned}$$

Theorem 3.3 *Let Δ^1, $\Delta^2 \in m(X)$ and $A, B \in I(X)$. Then followings hold :*
(1) $\Delta^1(A \bigcup B) = \Delta^1(A) \bigcup \Delta^1(B)$,
(2) $(\Delta^1 \bigcup \Delta^2)(A) = \Delta^1(A) \bigcup \Delta^2(A)$,
(3) $A \subset B \Rightarrow \Delta^1(A) \subset \Delta^2(B)$.

Theorem 3.4 *Let Δ be a binary relation on $I(X)$. Then Δ is a preproximity of IFSs on X if and only if $\Delta = \Delta^{-1}$ and $\Delta(A) \in \Gamma(X)$, $\forall\ A \in I(X)$.*

Definition 3.5 Let $\Delta \in m(X)$, $A \in I(X)$. Then $B \in I(X)$ is called a neighbourhood (in short nbd) of A with respect to Δ if $B^c \notin \Delta(A)$.
The collection of all nbds of A w.r.t. Δ is denoted by $N(\Delta, A)$.

Theorem 3.6 *Let $\Delta, \Delta^1, \Delta^2 \in m(X)$. Then followings hold :*
(1) $N(\delta, \tilde{0}) = I(X)$,
(2) if $B \in N(\Delta, A)$, $B^1 \in N(\Delta, A^1)$, *then* $B \bigcup B^1 \in N(\Delta, A \bigcup A^1)$,
(3) $N(\Delta, A \bigcup B) = N(\Delta, A) \bigcap N(\Delta, B)$,
(4) $N(\Delta, A) \subset N(\Delta, A^1)$ *if* $A^1 \subset A$.

(5) $N(\Delta^1 \bigcup \Delta^2, A) = N(\Delta^1, A) \bigcap N(\Delta^2, A)$,
(6) $N(\Delta^1, A) \subset N(\Delta^2, A)$ *if* $\Delta^2 \subset \Delta^1$.

Definition 3.7 Let $\Delta \in m(X)$, $A \in I(X)$. We define $C_\Delta : I(X) \to I(X)$ by

$$C_\Delta A = A \bigcup (\bigcup \{(p,q)_x : (p,q)_x \in \Delta(A)\})$$

where $(p,q)_x = \{< x, p, q >: x \in X\}$, $0 < p$, $0 \le q$ and $p + q \le 1$.
C_Δ is called the closure operator induced by Δ on X.

Theorem 3.8 *Let* $\Delta, \Delta^1 \in m(X)$. $A, B \in I(X)$. *Then* C_Δ *satisfies the following conditins :*
(1) $C_\Delta \tilde{0} = \tilde{0}$,
(2) $A \subset C_\Delta A$,
(3) $C_\Delta(A \bigcup B) = C_\Delta A \bigcup C_\Delta B$,
(4) $C_\Delta A \subset C_{\Delta^1}(A)$ if $\Delta \subset \Delta^1$.

From the above Theorem it is mentioned that C_Δ is a Čech closure operator and it is a Kuratowski closure operator if $C_\Delta(C_\Delta A) = C_\Delta A$, $\forall\ A \in I(X)$.

Theorem 3.9 *Let* Δ^1, $\Delta^2 \in m(X)$. *Then* $\forall\ p \in (0,1], \forall\ q \in [0,1)$ *with* $p + q \le 1$, $\forall\ x \in X, \Delta^1((p,q)_x) = \Delta^2((p,q)_x)$ *implies* $C_{\Delta^1} A = C_{\Delta^2} A$, $\forall\ A \in I(X)$.

Theorem 3.10 *For a proximity* π *of IFSs on* X, C_π *is a Kuratowski closure operator iff*

$$\tilde{1} \in \pi(C_\pi B) \Rightarrow \tilde{1} \in \pi(B).$$

Definition 3.11 Let $\Delta \in m(X)$ and $F \in \phi(X)$. Then we define

$$\Delta(F) = \bigcap \{\Delta(A) : A \in F\}.$$

Theorem 3.12 *For* $\Delta, \Delta^1, \Delta^2 \in m(X)$ *and* $F, F^1, F^2 \in \phi(X)$ *followings hold :*
(1) $\Delta(F) \in \Gamma(X)$,
(2) $\Delta(A) = \bigcup\{\Delta(V) : V \in \omega(X),\ A \in V\}$,
(3) $F^1 \subset \Delta(F^2) \Rightarrow F^2 \subset \Delta(F^1)$,
(4) $(\Delta^1 \bigcup \Delta^2)(F) = \Delta^1(F) \bigcup \Delta^2(F)$,
(5) $\Delta(F^1 \bigcap F^2) = \Delta(F^1) \bigcup \Delta(F^2)$.

Theorem 3.13 *For a proximity* π *of IFSs on* X, $F \subset \pi(F)$, $\forall$ *proper filter* F *of IFSs on* X.

Definition 3.14 Let $\Delta \in m(X)$. A subfamily T of $I(X)$ is said to be Δ-compatible if

$$A, B \in T \Rightarrow A \in \Delta(B).$$

A Δ-compatible grill is called a Δ-clan.

Theorem 3.15 *For $\Delta \in m(X)$, $G \in \Gamma(X)$, the followings are equivalent :*
(1) G is a Δ-clan,
(2) If $V \in \omega(X)$ such that $V \subset G$ then $G \subset \Delta(V)$,
(3) $G \subset \bigcap\{\Delta(V) : V \in \omega(X), V \subset G\}$,
(4) If $V^1, V^2 \in \omega(X)$ such that $V^1, V^2 \subset G$ then $V^1 \subset \Delta(V^2)$.

Theorem 3.16 *Let $\Delta \in m(X)$. Then every Δ-clan is contained in a maximal Δ-clan.*

Lemma 3.17 Let $\Delta \in m(X)$. If $A \in \Delta(B)$, then there exists $V^1, V^2 \in \omega(X)$ such that $A \in V^1$, $B \in V^2$ and $V^1 \subset \Delta(V^2)$.

Theorem 3.18 *Let $\pi \in M(X)$. If $A \in \pi(B)$, then there is a π-clan of the form $V^1 \bigcup V^2$ where V^1, $V^2 \in \omega(X)$ such that $A \in V^1$ and $B \in V^2$.*

Corollary 3.19 Let $\pi \in M(X)$. If $A \in \pi(B)$, then there exists a maximal π-clan containing $\{A, B\}$.

Corollary 3.20 Let $\pi \in M(X)$. Then
$\pi = \bigcup\{G \times G : G \text{ is maximal } \pi\text{-clan }\}$
$= \bigcup\{G \times G : G \text{ is a } \pi\text{-clan }\}$.

Remark 3.21 It is known that one of the most fundamental results in the area of proximities of fuzzy sets is that they are clan generated structure [3]. Because of the above representation, it follows that proximities of IFSs are also clan generated structures in the sense of their description as in the above Theorem.

References

(1) K.Atanassov and S.Stoeva, Intuitionistic fuzzy sets, Proc. of Polish Symp. on Interval and Fuzzy Mathematics, Poznan (Aug. 1983) 23-26.

(2) K.Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20 (1) (1986) 87-96.

(3) K.Atanassov, Review and new results on intuitionistic fuzzy sets, Preprint IM-MFAIS-1-88,

Sofia, 1988.

(4) K.Atanassov, New operators defined over the intuitionistic fuzzy sets, Fuzzy Sets and Systems, 61 (2) (1993) 131-142.

(5) P.Burillo, H.Bustince, Two operators on interval-valued intuitionistic fuzzy sets : Part I, Comptes rendus de l'Acade'mie bulgare des Sciences, Tome 47, N°12, 1994.

(6) P.Burillo, H.Bustince, Two operators on interval-valued intuitionistic fuzzy sets : Part II, Comptes rendus de l'Acade'mie bulgare des Sciences, Tome 48, N°1, 1995.

(7) K.C.Chattopadhyay, U.K.Mukherjee, S.K.Samanta, Role of clans in the Proximities of fuzzy sets, The Journal of Fuzzy Mathematics 4 (1996), No-3.

(8) Doğan Çoker, An introduction to intuitionistic fuzzy topological spaces, Fuzzy Sets and Systems 88 (1997) 81-89.

(9) Doğan Çoker, An introduction to Fuzzy Subspaces in Intuitionistic Fuzzy Topological Spaces, The Journal of Fuzzy Mathetatics Vol.4 No.4, 1996 749-764.

(10) S.K.Samanta and Tapas Kumar Mondal, Intuitionistic gradation of openness : Intuitionistic fuzzy topology, Communicated to "Fuzzy Sets and Systems".

(11) Tapas Kumar Mondal and S.K.Samanta, Topology of interval-valued intuitionistic fuzzy sets, Communicated to "Fuzzy Sets and Systems".