The Refinement of Syntopogenous Structure on Completely Distributive Lattice*

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Abstract:

In this paper, first the concept of refinement of semi-topogenous order on completely distributive lattice has been introduced. Second, the refine structure of syntopogenous structure on completely distributive lattice has been resarched. Some important properties have been obtained. And further, the extension of syntopoenous structure on completely distributive lattice will be discussed.

keywords:

Topology, topogenous order, syntopogenous structure, lattice.

1 Preliminaries

In this paper, L will be a completely distributive lattice. 1 and 0 respectively maximal and minimal elements of L.

Definition 1.1[1,2]. A binary relation $\ll$ on L is called a semi-topogenous order if it satisfies the following axioms: (1) $0\ll 0$ and $1\ll 1$; (2) $a\ll b$ implies $a\leqslant b$; (3) $a_1\leqslant a\ll b\leqslant b_1$ implies $a_1\ll b_1$.

A semi-topogenous order is called

(I) topogenous: if $a\ll b$ and $c\ll d$ implies $a\vee c\ll b\vee d$ and $a\wedge c\ll b\wedge d$;

(II) perfect: if $a_i\ll b_i$ $i\in I$ implies $\vee a_i\ll\vee b_i$;

(III) co-perfect: if $a_i\ll b_i$ $i\in I$ implies $\wedge a_i\ll\wedge b_i$;

(IV) biperfect: if it is perfect and co-perfect;

(V) symmetrical: if $\ll=\ll^c$. Where L with order-reversing involution "$'$" and $a\ll^c b$ iff $b'\ll a'$. A semi-topogenous order $\ll_2$ is called finer than another one $\ll_1$ (i. e. $\ll_1$ is called coarser than $\ll_2$) if $a\ll_1 b$ implies $a\ll_2 b$, denoted by $\ll_1\leqslant\ll_2$. If $\ll_1\leqslant\ll_2$ and $\ll_2\leqslant\ll_1$, then $\ll_1,\ll_2$ are called equivalent, denoted by $\ll_1=\ll_2$.

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Definition 1.2[1,2]. A syntopogenous structure on L is a nonempty family S of topogenous orders on L having the following two properties: (S_1) For any $\ll_1, \ll_2 \in S$, there exists $\ll \in S$ such that $\ll_1 \vee \ll_2 \leqslant \ll$. ($S_2$) For any $\ll \in S$, there exists $\ll_1 \in S$ such that $\ll \leqslant \ll_1^2$.

(L,S) is called a syntopogenous space. A syntopogenous structure S_2 is called finer than another one S_1 (i.e. S_1 is coarser than S_2), if for any $\ll_1 \in S_1$ there exists $\ll_2 \in S_2$ finer than $\ll_1$, denoted by $S_1 \leqslant S_2$. If $S_1 \leqslant S_2$ and $S_2 \leqslant S_1$, then we call S_1 and S_2 equivalent, denoted by $S_1 = S_2$.

Relative notions and signals please see [3,4,5].

2 The concept of refinement of semi-topogenous order on L

Definition 2.1. Let $\ll$ be a semi-topogenous order on L and $h \in L$. We consider a binary relation $\ll *h$ on L as follows:

$a \ll *h\, b$ iff there exists $c \in L$ such that $a \ll c$ and $a \vee (c \wedge h) \leqslant b$.

Theorem 2.1. If $\ll$ be a semi-topogenous order on L, then $\ll *h$ is a semi-topogenous order on L.

Proof. (1)Choose $c = 0$, then $0 \ll 0$ and $0 \vee (0 \wedge h) \leqslant 0$ so $0 \ll *h\, 0$; similarly, choose $c = 1$, then $1 \ll 1$ and $1 \vee (1 \wedge h) \leqslant 1$ so $1 \ll *h\, 1$.

(2) If $a \ll *h\, b$ then there exists $c \in L$ such that $a \ll c$ and $a \vee (c \wedge h) \leqslant b$. then $a \leqslant a \vee (c \wedge h) \leqslant b$, so $a \leqslant b$.

(3) If $a_1 \leqslant a \ll *h\, b \leqslant b_1$, then there exists $c \in L$ such that $a \ll c$ and $a \vee (c \wedge h) \leqslant b$, so $a_1 \ll c$ and $a_1 \vee (c \wedge h) \leqslant b \leqslant b_1$, thus $a_1 \ll *h\, b_1$.

Theorem2.2. Let $\ll$ be a semi-topogenous order on L, then (1)$\ll *0 = \leqslant$; (2)$\ll *1 = \ll$.

Proof. (1)If $a \ll *0\, b$ iff there exists $c \in L$ such that $a \ll c$ and $a \vee (c \wedge 0) \leqslant b$, then $a \vee (c \wedge 0) = a \leqslant b$; conversely, if $a \leqslant b$, choose $c = 1 \in L$, then $a \ll 1$ and $a \vee (1 \wedge 0) = a \leqslant b$, so $a \ll *0\, b$.

(2)If $a \ll *1\, b$ iff there exists $c \in L$ such that $a \ll c$ and $a \vee (c \wedge 1) \leqslant b$, then $a \ll c \leqslant b$ so $a \ll b$; conversely, if $a \ll b$, choose $c = b \in L$ then $a \ll b$ and $a \vee (b \wedge 1) = b \leqslant b$, thus $a \ll *1b$.

Theorem2.3. Let $\ll, \ll_1$ be semi-topogenous order on L and $h, h_1 \in L$, then (1)If $\ll \leqslant \ll_1$ implies $\ll *h \leqslant \ll_1 *h$; (2)If $h \leqslant h_1$ implies $\ll *h_1 \leqslant \ll *h$; (3)$\ll \leqslant \ll *h \leqslant \leqslant$ (for any $h \in L$).

Proof (1)If $a \ll *h\, b$ iff there exists $c \in L$ such that $a \ll c$ and $a \vee (c \wedge h) \leqslant b$, then $a \ll_1 c$ and $a \vee (c \wedge h) \leqslant b$, so $a \ll_1 *h\, b$.

(2) If $a \ll *h_1 b$ iff there exists $c \in L$ such that $a \ll c$ and $a \vee (c \wedge h_1) \leqslant b$, then $a \ll c$ and $a \vee (c \wedge h) \leqslant a \vee (c \wedge h_1) \leqslant b$, so $a \ll *h\, b$.

(3) Since $0 \leqslant h \leqslant 1$ (for any $h \in L$), by (2) we get $\ll *1 \leqslant \ll *h \leqslant \ll *0$, so $\ll \leqslant \ll *h \leqslant \leqslant$ (for any $h \in L$), *by Theorem* 2.2.

Let $H = \{\ll * h; h \in L\}$ is a family of refinement of semi-topogenous order on L. It is easy to see that $\leqslant$ and $\ll$ are respectively maximal and minimal elements of H. Now we define respectively $\bigcup_{h \in L} \ll * h$ and $\bigcap_{h \in L} \ll * h$ as follows :

$a \bigcup_{h \in L} \ll * h\, b$ iff there is $h_1 \in L$ such that $a \ll * h_1 b$; Moreover, $a \bigcap_{h \in L} \ll * h\, b$ iff for any $h \in L$ having $a \ll * h\, b$.

It is easy to see that $\bigcup_{h \in L} \ll * h$ and $\bigcap_{h \in L} \ll * h$ both are elements of H. Summarizly, we get

Theorem2. 4. Let $H = \{\ll * h; h \in L\}$ be a family of refinement of semi-topogenous order on L, then H is a perfect lattice.

3 The properties of refinement of semi-topogenous order on L

Theorem3. 1. Let $\ll$ be a semi-topogenous order on L, then $(\ll * h)^q = \ll^q * h$.

Proof. $a(\ll * h)^q b$ iff there exist $a_i, b_j \in L (i = 1,2,\cdots m, j = 1,2,\cdots,n)$ such that $a = \vee_{i=1}^m a_i, b = \wedge_{j=1}^n b_j, a_i \ll * h\, b_j$ for any i, j iff there exist $c_{ij} \in L$ such that $a = \vee_{i=1}^m a_i, b = \wedge_{j=1}^n b_j$, $a_i \ll c_{ij}$ and $a_i \vee (c_{ij} \wedge h) \leqslant b_j$; for any $i. j$. Iff choose $c = \wedge_{i=1}^m \wedge_{j=1}^n c_{ij}$, we can obtain $a \ll^q c$ and $a \vee (c \wedge h) \leqslant b$. Iff $a \ll^q * h\, b$.

Theorem3. 2. Let $\ll$ be a semi-topogenous order on L, then $\ll^2 * h \leqslant (\ll * h)^2$.

Proof. $a \ll^2 * h\, b$ iff there exists $c \in L$ such that $a \ll^2 c$ *and* $a \vee (c \wedge h) \leqslant b$ iff there exists $d \in L$ such that $a \ll d \ll c$ *and* $a \vee (c \wedge h) \leqslant b$, then choose $c_1 = a \vee (d \wedge h)$, since $a \ll d$ and $a \vee (d \wedge h) = c_1 \leqslant c_1$, so $a \ll * hc_1 \cdots$① since $c_1 = a \vee (d \wedge h) \leqslant d \ll c$ and $c_1 \vee (c \wedge h) = a \vee (d \wedge h) \vee (c \wedge h) = a \vee (c \wedge h) \leqslant b$. (i. e. $c_1 \ll c$ and $c_1 \vee (c \wedge h) \leqslant b$), so $c_1 \ll * h\, b \cdots$②. Thus $a(\ll * h)^2 b$, *by* ① *and* ②.

Theorem3. 3. Let $\ll$ be a semi-topogenous order on L, then (1) $(\ll * h)^p = \ll^p * h$; (2) $(\ll * h)^b = \ll^b * h$.

Proof (1) $a \ll^p * h\, b$ iff there exists $c \in L$ such that $a \ll^p c$ and $a \vee (c \wedge h) \leqslant b$ iff there exist $\{a_i, i \in I\}$ such that $a = \vee_{i \in I} a_i, a_i \ll c$ and $a \vee (c \wedge h) \leqslant b$, for any $i \in I$ iff there exist $\{a_i, i \in I\}$ such that $a = \vee_{i \in I} a_i, a_i \ll c$ and $a_i \vee (c \wedge h) \leqslant b$, for any $i \in I$ iff there exist $\{a_i, i \in I\}$ such that $a = \vee_{i \in I} a_i, a_i \ll * h\, b$, for any $i \in I$. i. e. $a(\ll * h)^p b$.

(2) $a(\ll * h)^b b$ iff there exist $\{a_i, i \in I\}$ and $\{b_j, j \in J\}$ such that $a = \vee_{i \in I} a_i, b = \wedge_{j \in J} b_j, a_i \ll * hb_j$, for any $i \in I, j \in J$iff there exist c_{ij} such that $a = \vee_{i \in I} a_i, b = \wedge_{j \in J} b_j, a_i \ll c_{ij}$ and $a_i \vee (c_{ij} \wedge h) \leqslant b$, for any $i \in I, j \in J$. Choose $c_j = \vee_{i \in I} c_{ij}$, then $a = \vee_{i \in I} a_i, b = \wedge_{j \in J} b_j, a_i \ll c_j$ and $a_i \vee (c_j \wedge h) \leqslant b_j$, for any $i \in I, j \in J$. Set $c = \wedge_{j \in J} c_j$, then $a = \vee_{i \in I} a_i, b = \wedge_{j \in J} b_j, a \ll^b c$ and $a \vee (c \wedge h) \leqslant b$, so $a \ll^b * h\, b$. Conversely: $a \ll^b * h\, b$ iff there exists $c \in L$ such that $a \ll^b c$ and $a \vee (c \wedge h) \leqslant b$ iff there exist $\{a_i, i \in I\}$ and $\{c_j, j \in J\}$ such that $a = \vee_{i \in I} a_i, c = \wedge_{j \in J} c_j, a_i \ll c_j$, and $a \vee (c \wedge h) \leqslant b$ for any $i \in I, j \in J$. Choose $b_j = a \vee (c_j \wedge h)$, then $a = \vee_{i \in I} a_i, \wedge b_j = a \vee (c \wedge h) = b_1 \leqslant b, a_i \ll c_j$ and $a_i \vee (c_j \wedge h) \leqslant b_j$, for any $i \in I, j \in J$, then $a = \vee_{i \in I} a_i, b_1 = \wedge b_j, a_i \ll * h\, b_j$, for any $i \in I, j \in J$. So $a(\ll * h)^b b_1 \leqslant b$, thus $a(\ll * h)^b b$.

Theorem3. 4. Let f: $L_1 \to L_2$ be a GOH([2]), let $\ll$ be a semi-topogenous order on L_2, then $f^{-1}(\ll) \leqslant f^{-1}(\ll * h)$.

Proof. $a f^{-1}(\ll) b$ iff there exist $a_1, b_1 \in L_2$ such that $a_1 \ll b_1$ and $a \leqslant f^{-1}(a_1), f^{-1}(b_1) \leqslant b$, then there exist $a_1, b_1 \in L_2$ such that $a_1 \ll * h\, b_1$ and $a \leqslant f^{-1}(a_1), f^{-1}(b_1) \leqslant b$. So $a f^{-1}(\ll * h) b$.

4 · The properties of refinement of syntopogenous structure on L

Now by $L*$ we shall denote a nonempty subsets of L satisfies the following condition:
for any $a_1, a_2 \in L*$ implies $a \leqslant a_1 \wedge a_2$, for some $a \in L*$. (I)

Theorem 4. 1. For any syntopogenous structure S on L, $S * L* = \{\ll * h.\ \ll \in S, h \in L*\}$ is also a syntopogenous structure on L.

Proof. Each element of $S * L*$ is topogenous, because by $(\ll * h)^q = \ll^q * h = \ll * h$, (for evey $\ll \in S, h \in L*$). If $\ll_1, \ll_2 \in S, h_1, h_2 \in L*$, and $\ll_1 \cup \ll_2 \leqslant \ll \in S, h_1 \wedge h_2 \geqslant h \in L*$, then $(\ll_1 * h_1) \cup (\ll_2 * h_2) \leqslant \ll * h$. For any $\ll \in S, h \in L*$, there is $\ll_1 \in S$ with $\ll \leqslant \ll_1^2$, then $\ll * h \leqslant \ll_1^2 * h \leqslant (\ll_1 * h)^2$. So $S * L*$ is a syntopogenous structure on L, by *Def.* 1. 2.

Theorem4. 2. Let S is a syntopogenous structure on L, then (1) $S \leqslant S * L*$; (2) $S * \{0\} = \{\leqslant\}$; (3) $S * \{1\} = S$.

The proof is omitted.

Theorem4. 3. Let S is a syntopogenous structure on L, then (1) $(S * L*)^p = S^p * L*$; (2) $(S * L*)^b = S^b * L*$.

Proof. Using theorem 3. 3, we get theorem 4. 3 easily.

Corollary4. 4. If S is perfect or biperfect, then $S * L*$ is also .

Theorem4. 5. Let $f: L_1 \to L_2$ be a *GOH*, define: $f^{-1}(L*) = \{f^{-1}(h): h \in L*\}$, then (1) $f^{-1}(S) \leqslant f^{-1}(S * L*)$; (2) $f^{-1}(S * L*) \leqslant f^{-1}(S) * f^{-1}(L*)$.

Proof(1)Using theorem 3. 4 we can get it(2)For any $f^{-1}(\ll * h) \in f^{-1}(S * L*)$, $a f^{-1}(\ll * h) b$ iff there exist $a_1, b_1 \in L_2$ such that $a_1 \ll * h b_1$, $a \leqslant f^{-1}(a_1), f^{-1}(b_1) \leqslant b$ iff there exist $a_1, b_1, c_1 \in L_2$ such that $a_1 \ll c_1$ and $a_1 \vee (c_1 \wedge h) \leqslant b_1$, $a \leqslant f^{-1}(a_1), f^{-1}(b_1) \leqslant b$. Choose $c = f^{-1}(c_1)$, then $a f^{-1}(\ll) c$ *and* $f^{-1}[a_1 \vee (c_1 \wedge h)] \leqslant f^{-1}(b_1)$. Then $a f^{-1}(\ll) c$ and $f^{-1}(a_1) \vee [f^{-1}(c_1) \wedge f^{-1}(h)] \leqslant f^{-1}(b_1)$. So $a f^{-1}(\ll) c$ and $a \vee (c \wedge f^{-1}(h)) \leqslant f^{-1}(a_1) \vee [f^{-1}(c_1) \wedge f^{-1}(h)] \leqslant f^{-1}(b_1) \leqslant b$. Thus $a f^{-1}(\ll) * f^{-1}(h) b$. i. e. $a f^{-1}(\ll * h) b$ implies $a f^{-1}(\ll) * f^{-1}(h) b$.

Now, we prove following 3 lemmas so that we can prove theorem 4. 6.

Lemma 1. Let $f: L_1 \to L_2$ be a *GOH*, and f is injective, then for any $a \in L_1, f(a) \leqslant$

$(f(a'))'$.

Proof. Since f is a injective, then $f^{-1}f(a') = a'$. As $a' \leqslant a'$ then $f^{-1}f(a') \leqslant a'$. So $a \leqslant f^{-1}(f(a'))'$. Thus $f(a) \leqslant (f(a'))'$.

Lemma 2. Let $f: L_1 \to L_2$ is a GOH, then for any $h \in L_2, h \leqslant [f(f^{-1}(h'))]'$.

Proof. Since f is a GOH, then $ff^{-1}(h') \leqslant h'$ so $h \leqslant [ff^{-1}(h')]'$.

Lemma 3. Let $f: L_1 \to L_2$ is a GOH, and f is a injective, then if $f(a) \ll (f(c'))'$ and $f(a) \vee [(f(c'))' \wedge h] \leqslant (f(b'))'$ imply $af^{-1}(\ll * h)b$.

Proof Choose $a_1 = f(a), b_1 = (f(b'))', c_1 = (f(c'))'$. then $a_1 \ll c_1$, and $a_1 \vee (c_1 \wedge h) \leqslant b_1$, $a = f^{-1}f(a) = f^{-1}(a_1), b' = f^{-1}f(b')$ (since f is a injective). imply $a_1 \ll * h\ b_1, a \leqslant a = f^{-1}(a_1)$, $b \geqslant b = f^{-1}(f(b'))' = f^{-1}(b_1)$. i.e. $af^{-1}(\ll * h)b$.

Theorem 4. 6. Let $f: L_1 \to L_2$ is a GOH, and f is a injective, then $f^{-1}(S * L*) = f^{-1}(S) * f^{-1}(L*)$.

Proof. From theorem 4. 5(2), we only need to prove $f^{-1}(S) * f^{-1}(L*) \leqslant f^{-1}(S * L*)$.

For any $f^{-1}(\ll) * f^{-1}(h) \in f^{-1}(S) * f^{-1}(L*)$. If $af^{-1}(\ll) * f^{-1}(h)b$, then there exists $c \in L_1$ such that $af^{-1}(\ll)c$ and $a \vee (c \wedge f^{-1}(h)) \leqslant b$. As $af^{-1}(\ll)c$, then there exists $a_1, c_1 \in L_2$ such that $a_1 \ll c_1$ and $a \leqslant f^{-1}(a_1), f^{-1}(c_1) \leqslant c$, so $a_1 \ll c_1$and $f(a) \leqslant a_1, c' \leqslant f^{-1}(c_1')$. Hence $a_1 \ll c_1$and $f(a) \leqslant a_1, f(c') \leqslant c_1'$. iff $f(a) \leqslant a_1 \ll c_1 \leqslant (f(c'))'$ then $f(a) \ll (f(c'))'$. And since $a \vee (c \wedge f^{-1}(h)) \leqslant b$, $b' \leqslant a' \wedge (c' \vee f^{-1}(h'))$, then $f(b') \leqslant f(a') \wedge (f(c') \vee ff^{-1}(h'))$($f$ is a GOH). So $(f(a'))' \vee [(f(c'))' \wedge (ff^{-1}(h'))']\leqslant (f(b'))'$, thus $f(a) \vee [(f(c'))' \wedge h] \leqslant (f(b'))'$. From lemma 1,2,3, we get $af^{-1}(\ll * h)b$.

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