# The Transformation from Semantical Topologies to Syntactical Topologies in Fuzzy Propositional Logic System L*

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## Abstract

In this paper, the interesting new informations are discovered and proved that a kind of semantical toplolgies can be transformed as a kind of syntactical topologies via the operators ( )–(α–T(L*)), ( )–(α⁺–T(L*)), and ( )–T(L*) on the fuzzy propositional logic system L* of Wang Guojun.

**Keywords:** fuzzy propositional logic; semantical topology; syntactical topology; $\Phi_R$–(α–tautology); $\Phi_R$–($\alpha^+$–tautology); $\Phi_R$–tautology

*Let* $L^* = (F(S), L)$ *be a fuzzy propositional logic system, where* $F(S)$ *be a set of abstract fuzzy propositions, and a free algebra of type* $(\neg, \wedge, \vee, \rightarrow)$; $S$ *be a set of abstract atomic fuzzy propositions;* $L$ *be an evaluation lattice and a algebra of type* $(\neg, \wedge, \vee, \rightarrow)$; *the ordering* $\leqslant$ *in* $L$ *may be linear, or nonlinear.*

*Let* $R$ *be any implication operator in the evaluation lattice* $L$, $\Omega_R$ *the set of all the* $R-$*evaluation mappings on* $F(S)$, *where every R-evaluation* $v_R$ *be a* $(\neg, \wedge, \vee, \rightarrow)-$ *homomorphism*

$$v_R: \begin{cases} F(S) \rightarrow L, \\ A \rightarrow v_R(A) = \bar{f}(v_R(p_1), \cdots, v_R(p_t)). \end{cases}$$

*Let* $\Phi_R$, $\Psi_R$, $\sum_R$, $\cdots$ *be the subsets of* $\Omega_R$.

*Suppose that* $\alpha \in L - \{0\}, A \in F(S)$. *If* $v_R(A) \geqslant \alpha$ *holds for every* $R-$*evaluation* $v_R \in \Phi_R$, *then* $A$ *is said to be a* $\Phi_R - (\alpha -$ *tautology). For each* $\alpha \in L - \{0\}$, *the set of all the* $\Phi_R - (\alpha -$ *tautologies) in the fuzzy propositional logic system* $L^*$ *is denoted by* $\Phi_R - (\alpha - T(L^*))$. *Thus a collection of operators*

$$(\ ) - (\alpha - T(L^*)) \colon P(\Omega_R) \rightarrow P(F(S)), \ (\alpha \in L - \{0\})$$

*are obtained. We can prove that a kind of semantical topologies can be transformed as a kind of syntactical topologies via the operators*

$(\ )-(\alpha-T(L^*)),\ (\alpha\in L-\{0\})$.

**Proposition** 1. *Suppose that* $\Phi_R\subset\Psi_R$, *then*

$$\Psi_R-(\alpha-T(L^*))\subset\Phi_R-(\alpha-T(L^*)),$$

*i.e., the operator*

$$(\ )-(\alpha-T(L^*)):P(\Omega_R)\to P(F(S))$$

*is a inverse – ordering mapping.*

*Proof. Let* $A\in\Psi_R-(\alpha-T(L^*))$. *Then* $v_R(A)\geqslant\alpha$ *holds for every* $v_R\in\Psi_R$. *Particularly,* $v_R(A)\geqslant\alpha$ *holds for each* $v_R\in\Phi_R$. *Therefore* $A\in\Phi_R-(\alpha-T(L^*))$. *So*

$$\Psi_R-(\alpha-T(L^*))\subset\Phi_R-(\alpha-T(L^*)).$$

*This completes the proof.*

**Propositin** 2. *Suppose that* $\{\Phi_R^\theta\mid\theta\in\Theta\}\subset P(\Omega_R\}$. *Then*

$$(\bigcup_{\theta\in\Theta}\Phi_R^\theta)-(\alpha-T(L^*))=\bigcap_{\theta\in\Theta}(\Phi_R^\theta-(\alpha-T(L^*))),$$

*i.e., the operator* $(\ )-(\alpha-T(L^*))$: $P(\Omega_R)\to P(F(S))$ *is a* $\bigcup-\bigcap$ *mapping.*

**Proof**. *It is clear that* $\Phi_R^\theta\subset\bigcup_{\theta\in\Theta}\Phi_R^\theta$ *holds for every* $\theta\in\Theta$ . *It follows from Proposition 1 that*

$$(\bigcup_{\theta\in\Theta}\Phi_R^\theta)-(\alpha-T(L^*))\subset\Phi_R^\theta-(\alpha-T(L^*)),$$

*and hence*

$$(\bigcup_{\theta\in\Theta}\Phi_R^\theta)-(\alpha-T(L^*))\subset\bigcap_{\Delta\in\Theta}(\Phi_R^\theta-(\alpha-T(L^*))).$$

*On the other hand, suppose that* $A\in\bigcap_{\theta\in\Theta}(\Phi_R^\theta-(\alpha-T(L^*)))$, *then* $A\in\Phi_R^\theta-(\alpha-T(L^*))$ *holds for every* $\theta\in\Theta$ , *i.e.,* $v_R(A)\geqslant\alpha$ *holds for every* $\theta\in\Theta$ *and every* $v_R\in\Phi_R^\theta$. *Hence* $v_R(A)\geqslant\alpha$ *holds for every* $v_R\in\bigcup_{\theta\in\Theta}\Phi_R^\theta$, *i.e.,*

$$A\in(\bigcup_{\theta\in\Theta}\Phi_R^\theta)-(\alpha-T(L^*)).$$

*Therefore,*

$$\bigcap_{\theta\in\Theta}(\Phi_R^\theta-(\alpha-T(L^*)))\subset(\bigcup_{\theta\in\Theta}\Phi_R^\theta)-(\alpha-T(L^*)).$$

*This completes the proof.*

**Propostion**3. *Suppose that* $\{\Phi_R^\theta\mid\theta\in\Theta\ \}\subset P(\Omega_R)$, *and* $\bigcap_{\theta\in\Theta}\Phi_R^\theta\in\{\Phi_R^\theta\mid\theta\in\Theta\ \}$. *Then*

$$(\bigcap_{\theta\in\Theta}\Phi_R^\theta)-(\alpha-T(L^*))=\bigcup_{\theta\in\Theta}(\Phi_R^\theta-(\alpha-T(L^*))),$$

*i.e., the operator*

$$(\ )-(\alpha-T(L^*)):\ P(\Omega_R)\to P(F(S))$$

*is a conditional* $\bigcap-\bigcup$ *mapping.*

**Proof.** *It is clear that* $\bigcap_{\theta\in\Theta}\Phi_R^\theta\subset\Phi_R^\theta$ *holds for every* $\theta\in\Theta$ . *It follows from Proposition* 1 *that*

$$\Phi_R^\theta-(\alpha-T(L^*))\subset(\bigcap_{\theta\in\Theta}\Phi_R^\theta)-(\alpha-T(L^*)),$$

*and hence*

$$\bigcup_{\theta\in\Theta}(\Phi_R^\theta-(\alpha-T(L^*))\subset(\bigcap_{\theta\in\Theta}\Phi_R^\theta)-(\alpha-T(L^*)).$$

*On the other hand, since there exists a* $\theta_0\in\Theta$ *such that* $\bigcap_{\theta\in\Theta}\Phi_R^\theta=\Phi_R^{\theta_0}$, *so*

$$(\bigcap_{\theta\in\Theta}\Phi_R^\theta)-(\alpha-T(L^*))=\Phi_R^{\theta_0}-(\alpha-T(L^*))\subset\bigcup_{\theta\in\Theta}(\Phi_R^{\theta_0}-(\alpha-T(L^*))).$$

*Therefore,*

$$(\bigcap_{\theta\in\Theta}\Phi_R^\theta)-(\alpha-T(L^*))=\bigcup_{\theta\in\Theta}(\Phi_R^\theta-(\alpha-T(L^*))).$$

*This completes the proof.*

**Theorem** 1. *Suppose that* $\mathbf{T}$ *is a topology on the lattice* $P(\Omega_R)$, *and any finite subcollection* $\mathbf{H}$ *of* $\mathbf{T}$ *is closed under the operation* $\bigcap$.
*Let*

$$\mathbf{T}-(\alpha-T(L^*))=\{\Phi_R-(\alpha-(T(L^*))\mid\Phi_R\in\mathbf{T}\},$$

*and*

$$P_*(F(S))=\{\mathbf{A}\mid\Omega_R-(\alpha-T(L^*))\subset\mathbf{A}\subset F(S)\}.$$

*Then* $\mathbf{T}-(\alpha-T(L^*))$ *must be a cotopology on the sublattice* $P_*(F(S))$ *of the lattice* $P(F(S))$.

**Proof.** (1) *It is obvious that both the minimum element* $\Omega_R-(\alpha-T(L^*))$ *and the maximum element* $\emptyset-(\alpha-T(L^*))$ *belong to* $\mathbf{T}-(\alpha-T(L^*))$.

(2) *Taking arbitrarily a subcollection* $\{\Phi_R^\theta-(\alpha-T(L^*))\mid\theta\in\Theta\ ,\Phi_R^\theta\in\mathbf{T}\}$ *of* $\mathbf{T}-(\alpha-T(L^*))$. *Notice that* $\bigcup_{\theta\in\Theta}\Phi_R^\theta\in\mathbf{T}$,

*so it follows from Proposition* 2 *that*

$$\bigcap_{\theta\in\Theta}(\Phi_R^\theta-(\alpha-T(L^*)))=(\bigcup_{\theta\in\Theta}\Phi_R^\theta)-(\alpha-T(L^*))\in\mathbf{T}-(\alpha-T(L^*)).$$

(3) *Taking arbitrarily a finite subcollection* $\{\Phi_R^\theta - (\alpha - T(L^*)) \mid \theta\in\Theta \ , \Phi_R^\theta \in \mathbf{T}\}$ *of* $\mathbf{T} - (\alpha - T(L^*))$. *Since any finite subcollection of* $\mathbf{T}$ *is closed under the operation* $\bigcap$, *so*

$$\bigcap_{\theta\in\Theta}\Phi_R^\theta \in \{\Phi_R^\theta \mid \theta\in\Theta\}.$$

*Notice that* $\bigcap_{\theta\in\Theta}\Phi_R^\theta \in \mathbf{T}$, *then it follows from Proposition* 3 *that*

$$\bigcup_{\theta\in\Theta}(\Phi_R^\theta - (\alpha - T(L^*))) = (\bigcap_{\theta\in\Theta}\Phi_R^\theta) - (\alpha - T(L^*)) \in \mathbf{T} - (\alpha - T(L^*)).$$

*Combine* (1),(2), *and* (3) *together, we see that* $\mathbf{T} - (\alpha - T(L^*))$ *is indeed a cotopology on the lattice* $P_*(F(S))$. *This completes the proof.*

*We have similar informations about operators* $(\ ) - (\alpha^+ - T(L^*))$: $P(\Omega_R) \to P(F(S))$, *and* $(\ ) - T(L^*)$: $P(\Omega_R) \to P(F(S))$. *Where, for every* $\Phi_R \subset \Omega_R$, $\Phi_R - (\alpha^+ - T(L^*))$ *is the set of all the* $\Phi_R - (\alpha^+ - \textit{tautologies})$ *in the fuzzy propositional logic system* $L^*$; $\Phi_R - T(L^*)$ *the set of all the* $\Phi_R - \textit{tautologies}$ *in* $L^*$. *Let* $A \in F(S)$ *be a fuzzy proposition, if* $v_R(A) > \alpha$ *holds for every evaluation* $v_R \in \Phi_R$, *then A is said to be a* $\Phi_R - (\alpha^+ - \textit{tautology})$. *Similarly, if* $v_R(A) = 1$ *holds for every evaluation* $v_R \in \Phi_R$, *then A is said to be a* $\Phi_R - \textit{tautology}$.

**Proposition** 4. *Suppose that* $\Phi_R \subset \Psi_R$. *Then*

$$\Psi_R - (\alpha^+ - T(L^*)) \subset \Phi_R - (\alpha^+ - T(L^*)),$$

*i.e., the operator* $(\ ) - (\alpha^+ - T(L^*)) : P(\Omega_R) \to P(F(S))$ *is a inverse – ordering mapping.*

**Proposition** 5. *Suppose that*

$$\{\Phi_R^\theta \mid \theta\in\Theta\} \subset P(\Omega_R).$$

*Then*

$$(\bigcup_{\theta\in\Theta}\Phi_R^\theta) - (\alpha^+ - T(L^*)) = \bigcap_{\theta\in\Theta}(\Phi_R^\theta - (\alpha^+ - T(L^*))),$$

*i.e., the operator* $(\ ) - (\alpha^+ - T(L^*)) : P(\Omega_R) \to P(F(S))$ *is a* $\bigcup - \bigcap$ *mapping.*

**Proposition** 6. *Suppose that* $\{\Phi_R^\theta \mid \theta\in\Theta\} \subset P(\Omega_R)$, *and* $\bigcap_{\theta\in\Theta}\Phi_R^\theta \in \{\Phi_R^\theta \mid \theta$

$\in\Theta\}$. *Then* $(\bigcap_{\theta\in\Theta}\Phi_R^\theta)-(\alpha^+ - T(L^*)) = \bigcup_{\theta\in\Theta}(\Phi_R^\theta - (\alpha - T(L^*)))$, *i.e., operator* $(\ )-(\alpha^+ - T(L^*))$: $P(\Omega_R)\to P(F(S))$ *is a conditional* $\bigcap-\bigcup$ *mapping.*

**Theorem** 2. *Suppose that* **T** *is a topology on the lattice* $P(\Omega_R)$, *and any finite subcollection of* **T** *is closed under the operation* $\bigcap$. *Let*

$\mathbf{T}-(\alpha^+ - T(L^*)) = \{\Phi_R - (\alpha^+ - T(L^*)) | \Phi_R \in \mathbf{T}\}$,

*and* $P_{**}(F(S)) = \{\mathbf{B} \mid \Omega_R - (\alpha^+ - T(L^*)) \subset \mathbf{B} \subset F(S)\}$. *Then* $\mathbf{T}-(\alpha^+ - \mathrm{T}(\mathrm{L}^*))$ must be a cotopology on the sublattice $\mathrm{P}_{**}(\mathrm{F(S)})$ of the lattice P(F(S)).

**Proposition** 7.*Suppose that* $\Phi_R \subset \Psi_R$. *Then*

$$\Psi_R - T(L^*) \subset \Phi_R - T(L^*),$$

*i.e., operator*

$$(\ ) - T(L^*):\ P(\Omega_R)\to P(F(S))$$

*is a inverse — ordering mapping.*

**Proposition** 8. *Suppose that* $\{\Phi_R^\theta | \theta\in\Theta\} \subset P(\Omega_R)$.
*Then* $\bigcup_{\theta\in\Theta}\Phi_R^\theta - T(L^*) = \bigcap_{\theta\in\Theta}(\Phi_R^\theta - T(L^*))$, *i.e., operator*
$(\ )-T(L^*)$:$P(\Omega_R)\to P(F(S))$ *is a* $\bigcup-\bigcap$ *mapping.*

**Proposition** 9.*Suppose that* $\{\Phi_R^\theta \mid \theta\in\Theta\}\subset P(\Omega_R)$, *and* $\bigcap_{\theta\in\Theta}\Phi_R^\theta \in \{\Phi_R^\theta \mid \theta \in\Theta\}$. *Then* $(\bigcap_{\theta\in\Theta}\Phi_R^\theta) - T(L^*) = \bigcup_{\theta\in\Theta}(\Phi_R^\theta - T(L^*))$, *i.e., operator* $(\ )-T(L^*)$: $P(\Omega_R)\to P(F(S))$ *is a conditional* $\bigcap-\bigcup$ *mapping.*

**Theorem** 3. *Suppose that* **T** *is a topology, and any finite subcollec — tion of* **T** *is closed under the operation* $\bigcap$. *Let*

$$\mathbf{T} - T(L^*) = \{\Phi_R - T(L^*) | \Phi_R \in \mathbf{T}\},$$

*and*

$$P_{***}(F(S)) = \{B \mid \Omega_R - T(L^*) \subset B \subset F(S)\}.$$

*Then* $\mathbf{T} - T(L^*)$ *is a cotopology on the sublattice* $P_{***}(F(S))$ *of the lattice* $P(F(S))$.

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