CARISTI TYPE FUZZY HYBRID FIXED POINT THEOREMS IN MENGER PROBABILISTIC METRIC SPACE (Ⅱ)

SHI CHUAN

Department of Applied Mathematics
Nanjing University of Science & Technology
Nanjing, 210014, People's Republic of China

ABSTRACT: In this paper, the Caristi type fuzzy hybrid fixed point theorem in M-PM-space is considered. A new fuzzy hybrid fixed point theorem and a new common fuzzy hybrid fixed point theorem of sequences of fuzzy mappings in M-PM-space are obtained. These theorems improve and generalize the Caristis fixed point theorem and correspoinding recent important results.

KEY WORDS: Probabilistic metric space, Menger space, Caristi's fixed point theo rem, fuzzy hybrid fixed point, common fuzzy hybrid fixed point.
1991 **AMS SUBJECT CLASSIFICATION CODES**: 54H25, 47H10

1 PRELIMINARIES

Throughout this paper, we assume the (E, F, Δ) is a Menger probabilistic metric space (briefly M-PM-Space), where t-norm Δ satisfies the condition:

$$\lim_{x\to 1^{-1}} \Delta(x,y) = y, \forall\ y \in [0,1] \tag{1.1}$$

DEFINITION 1.1 a mapping $A: E \to [0,1]$ is called a fuzzy subset over E , we denote by $\mathscr{F}(E)$ the famity of all fuzzy subsets over E , a mapping $S: E \to \mathscr{F}(E)$ is called a fuzzy mapping over E. Let $S: E \to \mathscr{F}(E)$, $T: E \to E$, if $p \in E$ such that $Sp(p) = \max_{u\in E} Sp(u)$ and $Tp = p$, then p is called a fuzzy hybrid fixed point of S and T. Let $S_K: E \to \mathscr{F}(E)$ $(K = 1, 2, \cdots)$, $T: E \to E$, if $p \in E$ such that $(\bigcap_{K=1}^{+\infty} S_K p)(p) = \max_{u\in E}(\bigcap_{K=1}^{+\infty} S_K p)(u)$ and $Tp = p$, then p is called a common fuzzy hybrid fixed point of $\{S_K\}$ and T.

DEFINITION 1.2 Let (E, F, Δ) is a M-PM-space, $T: E \to E$, we say T

satisfies the condition (1. 2), if $\forall\ x, y \in E$,

$$F_{x,Ty}(t) \leqslant F_{Tx,Ty}(x), \forall\ t > 0 \tag{1.2}$$

2 MAIN RESULTS

Let $S: E \to \mathscr{F}(E)$ is a fuzzy mapping, $O: E \to (0,1]$ is a real-valued function, $\forall\ x \in E$ let $Gx = (Sx)_{O(x)} = \{u \mid Sx(u) \geqslant O(x)\}$, the $G: E \to 2^E$ is a set-valued mapping, throughout this paper we denote this mapping by G.

THEOREM 2. 1 Let (E, F, Δ) is a complete M-PM-space, Δ satisfies the condition (1,1), $\Phi, E \to (-\infty, +\infty]$ is a lower semicontinuous function, bounded from below, $\not\equiv +\infty$, $T: E \to E$ is continuous and satisfies the condition (1. 2), let $S: E \to \mathscr{F}(E)$ is a fuzzy mapping.

(i) If there exists a real-valued function $O: E \to (0,1]$ such that $\forall\ x \in E$, $Gx \neq \varnothing$, $TGx = GTx$, and $\exists\ y \in Gx$ such that

$$F_{y,Tx}(t) \geqslant H(t - (\Phi(x) - \Phi(y))), t > 0 \tag{2.1}$$

then for every $u \in E$ with $\Phi(Tu) \neq +\infty$ and $\beta > 1$, there exists $p \in E$, such that $p = Tp$ and $Sp(p) \geqslant O(p)$, moreover

$$F_{Tu,p}(t) \geqslant H(t - \beta(\Phi(Tu) - \Phi(p))) \tag{2.2}$$

if $\Phi(Tu) \leqslant \inf\limits_{x \in T(E)} \Phi(x) + \varepsilon < \inf\limits_{x \in T(E)} \Phi(x) + 1$, then p satisfies

$$F_{Tu,p}(t) \geqslant H(t - \sqrt{\varepsilon}) \tag{2.3}$$

(ii) In particular, if $\forall\ x \in E, O(x) = \max\limits_{u \in E} Sx(u)$ satisfies the conditions of (i), then there exists $p \in E$, p is a fuzzy hybrid fixed point of S and T, moreover satisfies (2. 2), (2. 3).

PROOF. Since $\forall\ x \in E, Gx = (Sx)_{O(x)} = \{u \mid Sx(u) \geqslant O(x)\}$, $G: E \to 2^E$ is a set-valued mapping, moreover satisifes the conditions of (i). By using $G: E \to 2^E \backslash \{\Phi\}$, we can define mapping $Q: E \to E$ such that $\forall\ x \in E, Qx = y$, where $y \in Gx$ and satisfies (2. 2). Thus $\forall\ x \in E$ which implies that

$$F_{Qx,Tx}(t) \geqslant H(t - (\Phi(x) - \Phi(Qx))), t > 0 \tag{2.4}$$

By Lemma 1. 2 of [9], $F(E) = T(E)$, moreover $T: E \to E$ is continuous, therefore $F(E)$ is a closed set, thus $(F(E), F, \Delta)$ is a complet M-PM-space, we prove that $\forall\ x \in F(E), Qx \in F(E)$, in fact, $\forall\ x \in F(E)$, since $x = Tx, GTx = TGx$, therefore $Qx = QTx \in GTx = TGx \subseteq T(E) = F(E)$, $\therefore Qx \in F(E)$.

Since $\Phi(Tu) \neq +\infty$, if $\Phi(Tu) = \inf_{x \in T(E)} \Phi(x)$, then by (2.4)

$$F_{QTu, TTu}(t) \geqslant H(t) - (\Phi(Tu) - \Phi(STu)), t > 0$$

We have $F_{QTu, TTu}(t) = H(t)$, so $QTu = TTu = Tu$, Taking $p = Tu \in F(E)$, we have $Tp = p = Qp \in Gp$, moreover satisfies (2.2), (2.3). If $\Phi(Tu) - \inf_{x \in T(E)} \Phi(x) = \varepsilon > 0$, since $(F(E), F, \Delta)$ is a complete M-PM-space, by Lemma 1.1 of [9] $\forall \lambda > 0, \exists p \in F(E)$ such that

$$F_{Tu,p}(t) \geqslant H(t - \frac{1}{\varepsilon\lambda}(\Phi(Tu) - \Phi(p))), \forall t > 0 \qquad (2.5)$$

$$F_{Tu,p}(t) \geqslant H(t - \frac{1}{\lambda}), \forall t > 0$$

$\forall x \in F(E), x \neq p, \exists t_0 = t_0(x) > 0$, such that

$$F_{x,p}(t_0) < H(t_0 - \frac{1}{\varepsilon\lambda}(\Phi(p) - \Phi(x))) \qquad (2.6)$$

we shall show that $Qp = p$. In fact, it $Qp \neq p$, then by (2.6), $\exists\ t_0 = t_0(Qp)$ such that

$$F_{Qp,p}(t_0) < H(t_0 - \frac{1}{\varepsilon\lambda}(\Phi(p) - \Phi(Qp)))$$

By (2.5) and taking $\lambda = \frac{1}{\varepsilon}$, we have

$$F_{Qp,p}(t_0) < H(t_0 - (\Phi(p) - \Phi(Qp))) \leqslant F_{Qp,Tp}(t_0) = F_{Qp,p}(t_0)$$

This is a contraction. So $Qp = p$. Thus $p \in F(E) = T(E)$ satisfies $Tp = p = Qp \in Gp$. By (2.5), we have

$$\begin{aligned} F_{Tu,p}(t) &\geqslant H(t - \frac{1}{\varepsilon\lambda}(\Phi(Tu) - \Phi(p)) \\ &= H(t - (\Phi(Tu) - \Phi(p))) \quad (\lambda = \frac{1}{\varepsilon}) \\ &\geqslant H(t - \beta(\Phi(Tu) - \Phi(p))) \quad \text{i.e.} \qquad (2.2) \end{aligned}$$

If $\Phi(Tu) \leqslant \inf_{x \in T(E)} \Phi(x) + \varepsilon < \inf_{x \in T(E)} \Phi(x) + 1$, tet $\beta = \frac{1}{\sqrt{\varepsilon}}$, we have

$$\begin{aligned} F_{Tu,p}(t) &\geqslant H(t - \beta(\Phi(Tu) - \Phi(p))) \\ &\geqslant H(t - \beta \cdot \varepsilon) \\ &\geqslant H(t - \sqrt{\varepsilon}) \quad \text{i.e.} \qquad (2.3) \end{aligned}$$

In particular, if $O(x) = \max_{u\in E} Sx(u)$ satisfies the conditions of (i), since $p = Qp \in Gp, Sp(p) \geqslant O(p) = \max_{u\in E} Sp(u) \geqslant Sp(p)$, $\therefore Sp(p) = \max_{u\in E} Sp(u)$, it is ob vions that p is a fuzzy hybrid fixed point of S and T.

THEOREM 2.2 Let (E,F,Δ), Φ, T satisfies the conditions of theorem 2.1 moreover $x_1 \neq x_2$ with $Tx_1 \neq Tx_2$, let $S_K: E \to \mathscr{F}(E)$ $(K = 1,2,\cdots)$ is a se quence of fuzzy mappings.

(i) If there exists a sequence of functions $O_K: E \to (0,1]$ $(K = 1,2,\cdots)$ such that $\forall\, x \in E, G_K x = (S_K x)_{O_K(x)} \neq \varnothing$, $TG_K x = G_K Tx$, and $\exists\, y \in G_K x$ $(K = 1,2,\cdots)$ such that

$$F_{y,Tx}(t) \geqslant H(t - (\Phi(x) - \Phi(y))), \forall\, t > 0 \tag{2.7}$$

Then for any $u \in E$ with $\Phi(Tu) \neq +\infty$ and $\beta > 1$, there exists $p \in E$ such that $p = Tp \in G_K p$ $(K = 1,2,\cdots)$ and satisfies (2.2), (2.3).

(ii) In particular, if $\forall\, x \in E, O_K(x) = \max_{u\in E} S_K x(u)$ $(K = 1,2,\cdots)$ satisfies the conditions of (i), then there exists $p \in E$, p is a common fuzzy hybrid fixed point of $\{S_K\}$ and T, moreover satisfies (2.2), (2.3).

PROOF, $\forall\, x \in E$, let $Gx = \bigcap_{K=1}^{+\infty} G_K x$, where $G_K x = (S_K x)_{O_K(x)}$ $(K = 1,2,\cdots)$, by the conditions of (i) $\forall\, x \in E, Gx \neq \varnothing$ and $\exists\, y \in G$ such that (2.7), by proof of Theorem 2.2 of [9] $TG = GT$, thus $G: E \to 2^E$ satisfies the conditions of theorem 2.1, for T and G applying theorem 2.1, we obtain $p \in E$, such that $p = Tp \in Gp$ and (2.2), (2.3), by $p \in Gp = \bigcap_{+\infty}^{+\infty} G_K p \subseteq G_K p$, $(K = 1,2,\cdots)$. In par ticular, if $O_K(x) = \max_{u\in E} S_K x(u)$ $(K = 1, 2, \cdots)$, by $S_K p(p) \geqslant O_K(p) = \max_{u\in E} S_K p(u) \geqslant S_K p(u)$, $\forall\, u \in E$, and $(\bigcap_{K=1}^{+\infty} S_K p)(p) = \min_K S_K p(p) \geqslant \min_K \max_{u\in E} S_K p(u) \geqslant \min_K S_K p(u) = (\bigcap_{K=1}^{+\infty} S_K p)(u)$, $\forall\, u \in E$, we have $(\bigcap_{K=1}^{+\infty} S_K p)(p) \geqslant \max_{u\in E}(\bigcap_{K=1}^{+\infty} S_K p)(u) \geqslant (\bigcap_{K=1}^{+\infty} S_K p)(p)$, thus $(\bigcap_{K=1}^{+\infty} S_K p)(p) = \max_{u\in E}(\bigcap_{K=1}^{+\infty} S_K p)(u)$, p is a common fuzzy hybrid fixed point of $\{S_K\}$ and T.

REMARK 2.1 For a complete metric space (E,d), we can define map ping $F: E \times E \to D$ as follow $F_{x,y}(t) = H(t - d(x,y))$, $t \in (-\infty, +\infty)$, it is easy to prove that $(E,F,\min)$ be a complete M-PM-space, $\Delta = \min$ satisfies the condition (1.1), therefore it is easy to prove that the theorems of this paper improve and generalize Carist's fixed point theorem and corresponding recent im portant results of [1—8].

REFERENCES

[1] Caristi. J. ,Fixed point theorems for mappings satisfying inwardrcess conditions,Trans. Amer. Math. Soc,215(1976)241—251.

[2] S. Z. Shi,Equivalence between Ekeland's Variational principle and Caristi's fixed point theorem,Adavn. Math. 16(2)(1987)203—206(in chinese)

[3] Chang S. S. ,Luo Q. ,Set-valued Carist's fixed point theorem and Ekelard's Variational prineiple,Appeled Mathematies and Mechanics,10(2)(1989) 111—113.

[4] Chang S. S. ,Huang N. J. ,Shi C. ,Set-Valued Carist's Theorem in PM-spaces,Tournal of Sichum University Natural Science Edition,Vol. 30. No. 1. (1993) (in Chinese)

[5] Jeong Sheok Ume,Some existence theorems generalizing fixed point theorems on complete metric spaces,Math. Japonica 40,No. 1(1994),109—114

[6] Chang S. S. Luo Q. ,Caristi's fixed point theorem for fuzzy mappings and Ekeland's Variational Principle,Fuzzy Sets and Systems 64 (1994) 119—125.

[7] Chang S. S. Yeol Te cho,Skin Min Kang,probabilistic matric spaces and nonlinear operator theory,sichnan university press (1994).

[8] Shi Chuan,Caristi type hybrid fixed point theorem in Menger probabilistic Metric Space,Appeled Mathematies and Mechanics,18(2)(1997)189—196.

[9] Shi C. Caristi type fuzzy hybrid fixed point theorem in menger probabilistic metric space,BUSEFAL,71 (1997),30—35.