The Properties of L-fuzzy Topological Sum Spaces

Leng Xue-Bin

Department of Mathematics, Liaocheng Teachers' University,
Shandong 252059, P. R. China

Abstract: The concept of L-fuzzy topological sum spaces is introduced by Meng. In this paper, some properties for L-fuzzy topological sum spaces are studied, and the relation between the L-fuzzy topological sum spaces and crisp topological sum spaces is exposed, and the stratum structures of L-fuzzy topological sum spaces are discussed.

Key Words: L-fuzzy topology; L-fuzzy topological sum space; Fuzzy lattice

1. Introduction

In crisp topology, topological sum space is a basic concept [4], and sum operation of topological spaces is of importance to simplify proofs and the description of examples. H. Ghanim et al. [3] have first extended the notion of crisp topological sum spaces to fuzzy topology. Meng have introduced the concept of L-fuzzy topological sum spaces in [1], and studied some additive L-fuzzy topological properties in [2]. In this paper, we discuss some properties of L-fuzzy topological sum spaces, and expose the relation between the L-fuzzy topological sum spaces and crisp topological sum spaces by taking the stratum structures of L-fuzzy topological spaces as the point of departure.

2. Preliminaries

Throughout this paper, L always denote a fuzzy lattice, i.e., a completely distributive and complete lattice with an order-reversing involution $'$, it's smallest element and greatest element are 0 and 1, respectively. Let X be a nonempty crisp set, and $A \subset X$, then χ_A denotes the characteristic function of A defined on X into $\{0,1\} \subset L$. A mapping from X into L is called an L-fuzzy set on X. The collection of all the L-fuzzy sets on X, denoted by L^X, can be naturally seen as a fuzzy lattice $(L^X, \leqslant, \wedge, \vee, ')$. The smallest element and the greatest element of L^X are 0_X and 1_X, respectively, where $0_X(x) \equiv 0$, and $1_X(x) \equiv 1$ for any $x \in X$. (L^X, δ) stands for an L-fuzzy topological space (L-fts, for short), where δ is a subfamily of L^X containing 0_X and 1_X, which is closed under finite intersection and arbitrary union operation. For $r \in L$ and $A \in L^X$, $A_{[r]} = \{x \in X: A(x) > r\}$, $l_r(A) = \{x \in X: A(x) \not\leqslant r\}$. $P(L) = \{r \in L: r$ is a prime element of L and $r \neq 1\}$. Let (L^X, δ) be an L-fts, and $r \in P(L)$. Then it is not difficult to prove (see [5]) both $l_r(\delta) = \{l_r(A): A \in \delta\}$ and $[\delta] = \{A \in \delta:$ A is crisp set$\}$ are crisp topologies on X. In addition, let $\varphi(\delta) = \{l_r(A): r \in L, A \in \delta\}$, then it is clear that $\varphi(\delta)$ is the subbase of some crisp topology on X, and the crisp topology is denoted by $l_L(\delta)$. An L-fts (L^X, δ) is called weak induced [6], if for each $A \in \delta$ and any $r \in L$, $\chi_{l_r(A)} \in \delta$.

Definition 2.1. Let $\Phi \neq Y \subset X$, $A \in L^Y$, $B \in L^X$. Then $A^*, A^{**} \in L^X$ and $B|Y \in L^Y$ are defined, respectively, as follows:

$$A^*(x) = \begin{cases} A(x), & x \in Y, \\ 0, & x \in Y, \end{cases} \quad \forall x \in X,$$

$$A^{**}(x) = \begin{cases} A(x), & x \in Y, \\ 1, & x \in Y, \end{cases} \quad \forall\, x \in X,$$

$$(B|Y)(y) = B(y), \quad \forall\, y \in Y.$$

Clearly, $A^{*}|Y = A = A^{**}|Y$.

Definition 2.2 [1]. Let $\{(L^{X_t}, \delta_t)\}_{t \in T}$ be a family of L-fts's. Put $X = \bigcup_{t \in T} X_t$. For each $t \in T$, $j_t: X_t \to X$ is crisp inclusion mapping (i. e., $j_t(x) = x$, for each $x \in X_t$), it naturally induces an L-fuzzy mapping $j_t: L^{X_t} \to L^X$. Then it is clear that $\delta = \{A \in L^X: \forall t \in T, j_t^{-1}(A) \in \delta_t\}$ is an L-fuzzy topology on X. δ is called the L-fuzzy sum topology of $\{\delta_t\}_{t \in T}$, and is denoted by $\sum_{t \in T} \delta_t$. L-fts (L^X, δ) is called L-fuzzy topological sum space of $\{(L^{X_t}, \delta_t)\}_{t \in T}$, and denoted by $\sum_{t \in T}(L^{X_t}, \delta_t)$.

Remark 2.3. In the sequel we will assume that the family of L-fts's described in definition 2.2 is pairwise disjoint, i. e., for $t, s \in T$ and $t \neq s$, $X_s \cap X_t = \Phi$. The reason is expounded in [1].

Proposition 2.4 [1]. Let $(L^X \delta) = \sum_{t \in T}(L^{X_t}, \delta_t)$. Then

(1) $\delta = \{A \in L^X: \forall\, t \in T,\ A|X_t \in \delta_t\}$;

(2) $B \in \delta'$ iff $\forall\, t \in T$, $B|X_t \in \delta_t'$;

(3) $\forall t \in T$, if $A_t \in \delta_t$ (resp. $A_t \in \delta_t'$), then $A_t^*, A_t^{**} \in \delta$ (resp. A_t^*, $A_t^{**} \in \delta'$);

(4) $\forall\, t \in T$, $\delta|X_t = \delta_t$.

3. The properties of L-fuzzy topological sum spaces

Theorem 3.1. Let $(L^X, \delta) = \sum_{t \in T}(L^X_t, \delta_t)$, then for each $r \in P(L)$,

$(X,\ l_r(\delta)) = \sum_{t\in T}(X_t,\ l_r(\delta_t))$.

Proof. For each $l_r(A) \in l_r(\delta)$, we want to prove that $\forall t \in T$, $l_r(A) \cap X_t \in l_r(\delta_t)$.

It is not difficult to check that

$$l_r(A) \cap X_t = l_r(A|X_t) \qquad (3.1)$$

It follows from $A \in \delta$ that $\forall t \in T$, $A|X_t \in \delta_t$.

Hence $l_r(A) \cap X_t = l_r(A|X_t) \in l_r(\delta_t)$. □

Theorem 3.2. Let $(L^X, \delta) = \sum_{t\in T}(L^{X_t}, \delta_t)$. Then $(X, [\delta]) = \sum_{t\in T}(X_t, [\delta_t])$.

Proof. For each $A \in [\delta]$, we have $\chi_A \in \delta$. Hence $\forall t \in T$, $\chi_A|X_t \in \delta_t$. It is not difficult to check that $\chi_A|X_t = \chi_{A\cap X_t}$. Then $\forall\ t \in T$, $\chi_{A\cap X_t} \in \delta_t$. Therefore $A\cap X_t \in [\delta_t]$. □

Theorem 3.3. $(L^X, \delta) = \sum_{t\in T}(L^{X_t}, \delta_t)$ is weak induced iff $\forall\ t \in T$, (L^{X_t}, δ_t) is weak induced.

Proof. Necessity. Suppose that (L^X, δ) is weak induced. $\forall t \in T$, $\forall\ A_t \in \delta_t$, $\forall r \in L$, we want to prove $\chi_{l_r(A_t)} \in \delta_t$. From $A_t \in \delta_t$ we see that $A_t^* \in \delta$, and so $\forall r \in L$, $\chi_{l_r(A_t^*)} \in \delta$. Therefore $\forall t \in T$, $\chi_{l_r(A_t^*)}|X_t \in \delta_t$. Also, it is not difficult to check that $\forall t \in T$,

$$\chi_{l_r(A_t^*)}|X_t = \chi_{l_r(A_t)}$$

This shows $\chi_{l_r(A_t)} \in \delta_t$.

Sufficiency. $\forall A \in \delta$, $\forall r \in L$, we want to prove $\chi_{l_r(A)} \in \delta$. From $A \in \delta$ we get that $\forall t \in T$, $A|X_t \in \delta_t$, and so $\forall r \in L$, $\chi_{l_r(A|X_t)} \in \delta_t$. It follows from proposition 2.4 (3) that $(\chi_{l_r(A|X_t)})^* \in \delta$. Thus $\bigvee_{t\in T}(\chi_{l_r(A|X_t)})^* \in \delta$. We next will prove that

$$\chi_{l_r(A)} = \bigvee_{t\in T}(\chi_{l_r(A|X_t)})^* \qquad (3.2)$$

First, $\forall r \in P(L)$, $\forall A_t \in L^{X_t}$, it is clear that

$$l_r(A_t) = l_r(A_t^*) \tag{3.3}$$

It is not difficult from Eq. (3.3) to check that

$$(\chi_{l_r(A_t)})^* = \chi_{l_r(A_t^*)}$$

Second, we will prove that $A = \bigvee_{t\in T}(A|X_t)^*$.

$\forall x \in X = \bigcup_{t\in T} X_t$, there exists $s \in T$ such that $x \in X_s$. Then

$$(\bigvee_{t\in T}(A|X_t)^*)(x) = \bigvee_{t\in T}(A|X_t)^*(x) = (A|X_s)(x) = A(x)$$

Hence $A = \bigvee_{t\in T}(A|X_t)^*$.

Finally, we have

$$\chi_{l_r(A)} = \chi_{l_r(\bigvee_{t\in T}(A|X_t)^*)} = \chi_{\bigcup_{t\in T} l_r((A|X_t)^*)}$$

$$= \bigvee_{t\in T}\chi_{l_r((A|X_t)^*)} = \bigvee_{t\in T}(\chi_{l_r(A|X_t)})^*$$

Therefore (3.2) holds. From this, $\chi_{l_r(A)} \in \delta$ follows immediately. □

Given a crisp topological space (X, τ), then $\chi_\tau = \{\chi_A : A \in \tau\}$ is clearly an L-fuzzy topology on X.

Theorem 3.4. $(X, \tau) = \sum_{t\in T}(X_t, \tau_t)$ iff $(L^X, \chi_\tau) = \sum_{t\in T}(L^{X_t}, \chi_{\tau_t})$.

Proof. It follows from $\forall A \subset X$, $\chi_{A\cap X_t} = \chi_A|X_t$. □

Lemma 3.5. Let (X, τ) and (X_t, τ_t), $t \in T$, be crisp topological spaces, and Ω be the subbase of (X, τ). Then $(X, \tau) = \sum_{t\in T}(X_t, \tau_t)$ iff $\forall A \in \Omega$, $\forall t \in T$, $A \cap X_t \in \tau_t$.

Proof. Necessity is clear. Let us prove the sufficiency.

$\forall B \in \tau$, assume that $B = \bigcup_{h\in H}(\bigcap_{i=1}^{k_h} A_i^h)$, where $A_i^h \in \Omega$. Then $\forall t \in T$, $B \cap X_t$

$$= \bigcup_{h\in H} \left(\bigcap_{i=1}^{k_h} (A_i^h \cap X_t) \right) \in \tau_t. \square$$

Theorem 3.6. Let $(L^X, \delta) = \sum_{t\in T} (L^{X_t}, \delta_t)$. Then

$$(X, l_L(\delta)) = \sum_{t\in T} (X_t, l_L(\delta_t)).$$

Proof. It follows from lemma 3.4 and Eq. (3.1). $\square$

Given an L-fts (L^X, δ), let $\lambda^*(\delta) = \{A \in \delta : \forall r \in L, \chi_{l_r(A)} \in \delta\}$. Then Zhang and Liu have proved in [6] that $\lambda^*(\delta)$ is an L-fuzzy topology on X.

Theorem 3.7. Let $(L^X, \delta) = \sum_{t\in T} (L^{X_t}, \delta_t)$. Then

$$(L^X, \lambda^*(\delta)) = \sum_{t\in T} (L^{X_t}, \lambda^*(\delta)).$$

Proof. $A \in \lambda^*(\delta) \Longrightarrow A \in \delta$, and $\forall r \in L$, $\chi_{l_r(A)} \in \delta$

$\Longrightarrow \forall t \in T$, $A|X_t \in \delta_t$, and $\forall r \in L$, $\chi_{l_r(A)} |X_t \in \delta_t$

$\Longrightarrow \forall t \in T$, $A|X_t \in \delta_t$, and $\forall r \in L$, $\chi_{l_r(A) \cap X_t} \in \delta_t$

$\Longrightarrow \forall t \in T$, $A|X_t \in \delta_t$, and $\forall r \in L$, $\chi_{l_r(A|X_t)} \in \delta_t$

$\Longrightarrow \forall t \in T, A|X_t \in \lambda^*(\delta_t)$ $\square$

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