On Lightly Compact Sets In L-Fuzzy Topological Spaces

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Abstract : In this paper we introduce the concept of L- fuzzy lightly compact set . The characterization of lightly compact sets are discussed .Also it is pointed out that lightly compactness is an " L- good extension" of usual light compactness.

Keywords :Fuzzy lattice , L - fuzzy topological spaces, lightly compact set, α -RF , r-cover , prime element ,fuzzy mapping.

1. Introduction

Lightly compactness is the important notion in topology and plays an important part in the crisp light compactness . Hence it is a spectacular problem how the notion is generalized in L- fuzzy topology. Chen Shuili introduced an almost F-compactness and L- fuzzy H-sets in L- fuzzy topological spaces in [2,3,4,5] . This paper will generalize the notions and the result of [7,8] ,and establish the theory of L- fuzzy topology.

2. Preliminaries

In this paper , $L = L(\leq, \vee, \wedge, ')$ always denotes a fuzzy lattice. 1 and 0 denote the greatest and the least elements of L respectively .Let X be a nonempty crisp set , L^x is the set of all L- fuzzy sets on X and M(L) the set of all nonzero irreducible elements of L .For $\Omega \subset L^x$, we define that $\Omega'=\{A':A\in\Omega\}$,$\vee\Omega=\vee\{A:A\in\Omega\}$, $\wedge\Omega = \wedge\{A : A \in \Omega\}$, $FS(\Omega) =\{\Psi \subset \Omega : \Psi$ is finite subfamily of $\Omega\}$.For $\alpha \in L$, $A_\alpha=\{x \in X : A(x) \geq \alpha\}$. For each $A \in L^x$,$\overline{A}$, A° , A' , will denote the closure, the interior and the pseudo - complement of A respectively ; A is called LF semiopen (LF semiclosed) if there exist a $B \in \delta$ ($G \in \delta'$) such that $B\subset A\subset \overline{B}$ ($G^\circ \subset A\subset G$); A is called LF regular open (closed) if $A=\overline{A}^\circ$ ($A= \overline{A^\circ}$). Let $SO(L^x)$ ($SC(L^x)$, $RO(L^x)$, $RC(L^x)$) be the family of LF semiopen (semiclosed,regular closed) sets in (L^x,δ).

Definition 2.1 [11] **a)** $p \in L$ is called a union-irreducible element of L, if for arbitrary $a,b \in L$ we have $p \leq a \vee b \Rightarrow p \leq a$ or $p \leq b$. The set of all the non-zero union-irreducible elements of L is denoted by M(L).

b) $p \in L$ is called a prime element of L if for arbitrary $a, b \in L$ we have

$a \wedge b \leq p \Rightarrow a \leq p$ or $b \leq p$.

Put $p(L) = \{ p \in L : p$ is prime element of L and $p \neq 1\}$. It is easy to check that $p \in M(L)$ iff $p^1 \in p(L)$.

Put $M(L,X) = \{ x_\alpha : x \in X , \alpha \in M(L) \}$, then we easily check that M(L,X) is just the set of all non-zero union-irreducible elements (molecule) of L^X.

According to Wang [11], in a completely distrubitive lattice, each element α has a greatest minimal set which we will denote by $\beta(\alpha)$. We will put $\beta^*(\alpha)=\beta(\alpha)\cap M(L)$ and $\alpha^*(r)=(\beta^*(r'))'$.

Definition 2.2 [3] Let (L^x,δ) be an L -fts, $A \in L^x$ and $\alpha \in M(L)$. $\Omega \subset \delta'$ is called an almost α-RF of A, if for each x_α in A, there is $P \in \Omega$ such that $P^\circ \in \eta(x_\alpha)$. Ω is called an almost α^--RF of A if there exists $\lambda \in \beta^*(\alpha)$ such that Ω is a λ-RF of A.

Definition 2.3 [3] Let (L^x,δ) be an L -fts, $A \in L^x$ and $r \in p(L)$.

(i) $\Omega \subset \delta$ is called an r-cover of A if for each $x \in X$, there is $B \in \Omega$ satisfying $B(x) \not\leq r$. Ω is called an r^+-cover of A if there exists $t \in \alpha^*(r)$ such that Ω is a t-cover of A.

(ii) $\Omega \subset \delta$ is called an almost r-cover of A if for each $x \in X$, there is $B \in \Omega$ satisfying $\overline{B}(x) \not\leq r$. $\Omega \subset \delta$ is called an almost r^+-cover of A if for each $x \in X$, there exists $t \in \alpha^*(r)$ such that Ω is an almost t-cover of A.

Definition 2.4 [9] Let (L^x, δ) be an L-fts, and $A \in L^x$. A is called strong fuzzy compact, if for each $\alpha \in M(L)$ and every α-RF Ω of A, there exists $\Psi \in FS(\Omega)$ such that Ψ is an α-RF of A.

Definition 2.5 [9] Let (L^x,δ) be an L-fts, $A \in L^x$, $r \in p(L)$, $\Omega \subset \delta$. Ω is called an r-cover of A, if for each $x \in A_{r'}$, there exists $U \in \Omega$ such that $U(x) \not\leq r$.

Definition 2.6 [9] Let $\Omega \subset L^x$, $r \in p(L)$. Ω is said to have finite r-intersection property (f. r- i.p., for short) in $A \in L^x$ if for each $\Psi \in FS(\Omega)$ there exists $x \in A_{r'}$ such that $(\wedge\Psi)(x) \geq r'$.

Definition 2.7 [10] Let (L^x,δ) be an L-fts, $A \in L^x$, $\Omega \subset L^x$, $\alpha \in L$. Ω is called the family which has almost α-intersection property (or briefly, almost f. α-i.p.) in A, if for each $\Psi \in FS(\Omega)$ there is $x \in A_\alpha$ such that $(\wedge\Psi^\circ)(x) \geq \alpha$. Ω is called the family which has almost f. α^+-i.p. in A, if for each $t \in \beta^*(\alpha)$, Ω has almost f. t-i.p. in A.

Definition 2.8 [5] Let (L^x,δ) be an L-fts. $B \in L^x$ is called LF regular semiopen if there exists a $Q \in RO(L^x)$ satisfying $Q \subset B \subset \overline{Q}$; B is called LF regular semiclosed if there exists a $P \in RC(L^x)$ with $P^\circ \subset B \subset P$.

Definition 2.9 [1] Let (L^x,δ) and (L^y, ε) be two L-fts and f: $L^x \rightarrow L^y$ a fuzzy mapping

(i) f is semicontinuous if $f^{-1}(B) \in SO(L^x)$ for each $B \in \varepsilon$.

(ii) f is almost continuous if $f^{-1}(B) \in \delta$ for each $B \in RO(L^y)$.

(iii) f is weakly continuous if $f^{-1}(B) \subset (f^{-1}(\overline{B}))^\circ$ for each $B \in \varepsilon$.

3. Lightly compact sets

Definition 3.1. Let (L^x,δ) be an L-fts, $A \in L^x$. A is called lightly compact, if for each countably α-RF Ω of A, there exists $\Psi \in FS(\Omega)$ such that Ψ is an almost α^--RF of A. Particularly, when 1_x is lightly compact set, we call (L^x,δ) as a lightly compact space.

Theorem 3.2. An L-fuzzy set A in (L^x,δ) is a lightly compact set iff for any $r \in p(L)$ and every countably r-cover Ω of A, there exists $\Psi \in FS(\Omega)$ such that Ψ is an almost r^+-cover of A.

Theorem 3.3 Let (L^x,δ) be an L-fts, $A \in L^x$. Then A is lightly compact set iff for each $r \in M(L)$ and every countably family $\Omega \subset \delta'$ which has the f. r^+-i.p. in A, there exists $x \in A_r$ such that $(\wedge\overline{\Omega})(x) \geq r$.

Theorem 3.4. Let (L^x,δ) be an L-fts, $A \in L^x$. Then A is ligtly compact set iff for each regular closed countably α-RF Ω of A, there exists $\psi \in FS(\Omega)$ such that ψ is an α^--RF of A.

Theorem 3.5. Let $(L^x,w_L(\tau))$ be the L-fuzzy topological space topologically generated by a crisp space (X,τ). Then $(L^x,w_L(\tau))$ is L-fuzzy lightly compact iff (X,τ) is lightly compact.

Proof. Assume that $(L^x,w_L(\tau))$ is L-fuzzy lightly compact space and Ω is an countably open cover of X. Put $\chi_\Omega=\{\chi_A: A \in \Omega\} \subset w_L(\tau)$. For each $r \in p(L)$, χ_Ω is a countably r-cover of 1_X. Indeed, for each $x \in X$, there exists $A \in \Omega$ such that $x \in A$, and so

$\chi_A(x)=1\not\leq r$. Hence χ_Ω is a countably r-cover of 1_X. From the Theorem 3.2, Ω has a finite subfamily $\{A_1,\ldots,A_n\}$ such that $\psi=\{\chi_{A_i}:i=1,2,\ldots,n\}$ is a finite almost r^+-cover of 1_X, i.e., for each $x\in X$, there is $\chi_{A_i}\in\psi$ satisfying $(\chi_{A_i})^-(x)\neq 0$. Since $(\chi_{A_i})^-=\chi_{\overline{A_i}}$ and $x\in\overline{A_i},\bigcup_{i=1}^{n}\overline{A_i}=X$. This implies that (X,τ) is lightly compact space.

Conversely, for each $r\in p(L)$, suppose that $\phi\subset w_L(\tau)$ is any countably r-cover of 1_X. Then for each $x\in X$, there exists $P\in\phi$ such that $P(x)\not\leq r$.i.e., $x\in l_r(P)\in\tau$. Hence $l_r(\phi)=\{l_r(P):P\in\phi\}\subset\tau$ is a countably cover of (X,τ). From the lightly compactness of (X,τ), there exists $\psi\in FS(\phi)$ such that $\overline{l_r(\phi)}$ is a cover of (X,τ). Hence for each $x\in X$, there exists $P\in\psi$ such that $x\in\overline{l_r(P)}\in\overline{l_r(\psi)}$,i.e., $\overline{P}(x)\not\leq r$. Hence ψ is an almost r^+-cover of 1_X. Therefore $(L^X,w_L(\tau))$ is L-fuzzy lightly compact.

Theorem 3.6. An L-fuzzy set A in (L^X,δ) is a lightly compact set iff every countably α-RORF Ω of A has a finite subfamily ψ that is an α^--RORF of A for each $\alpha\in M(L)$.

Definition 3.7. Let (L^x,δ) be an L-fts, $A\in L^x$. A is called RS-compact, if for each $\alpha\in M(L)$ and each α-RSORF Ω of A, there exists $\psi\in FS(\Omega)$ such that ψ is an α-RSORF of A. (L^x,δ) is called RS-compact, if 1_x is RS-compact.

It is easy to verify that crisp RS-compactness is a special case of fuzzy RS-compactness.

Theorem 3.8. Let (X,τ) be a crisp topological space and (L^x,δ) be the corresponding L-fts, then (X,τ) is RS-compact iff $(L^x, w(\tau))$ is RS-compact.

Theorem 3.9. An L-fuzzy set A in L-fts (L^x,δ) is RS-compact iff for each $r\in p(L)$ and every r-regular semiopen cover Ω of A, there exists $\psi\in FS(\Omega)$ such that ψ is a r-regular semiopen cover of A.

Theorem 3.10. Let (L^x,δ) and (L^y,ε) be L-fts and $f: (L^x,\delta)\rightarrow(L^y,\varepsilon)$ be an almost continuous L-fuzzy mapping. If A is an L-fuzzy lightly compact set in L^x, then f(A) is an L-fuzzy lightly compact set in L^y.

Theorem 3.11. If $f: (L^x,\delta)\rightarrow(L^y,\varepsilon)$ is both weakly continuous and semicontinuous fuzzy mapping and A is an lightly compact set in (L^x,δ), then f(A) is lightly compact set.

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