ON INTERVAL-VALUED FUZZY NUMBERS

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Abstract: Bounded closed interval-valued fuzzy numbers are introduced based on interval-valued fuzzy sets, and their algebraic operations are discussed. The degree of uncertainty of the unmbers is portraied from the view of information theory. The concept of induced fuzzy numbers is presented, and the lattice theory structure of the family of induced fuzzy numbers is pointed out.

Keywords: Bounded closed interval-valued number, interval-valued fuzzy set, induced fuzzy number, uncertainty.

1. Preliminaries

In this paper, I and R stand for real unit closed interval [0, 1] and the set of all real numbers, respectively, and $[I]=\{[a^-,a^+]: a^-\le a^+, a^-, a^+\in I\}$, $[R]=\{[a,b]: a\le b,\ a,b\in R\}$.

Definition 1.1 For $[a_t, b_t]$, $[a_i, b_i]\in[R]$, $t\in T$, $i=1,2$.

(1) $$\bigvee_{t\in T}[a_t, b_t]=[\bigvee_{t\in T} a_t, \bigvee_{t\in T} b_t],$$

$$\bigwedge_{t\in T}[a_t, b_t]=[\bigwedge_{t\in T} a_t, \bigwedge_{t\in T} b_t].$$

(2) $[a_1,b_1]\leq[a_2,b_2]$ iff $a_1 \leq a_2$ and $b_1,\leq b_2$,
$[a_1,b_1]=[a_2,b_2]$ iff $a_1 = a_2$ and $b_1 = b_2$.

Clearly , $([R],\leq,\vee,\wedge)$ forms a lattice.

Definition 1.2 Let X be a non-empty set. A mapping A: X→[I] is called interval-valued fuzzy set on X. The set of all interval-valued fuzzy sets on X is denoted by $[I]^X$.

For $A\in[I]^X$, assuming that $A(x)=[A^-(x),A^+(x)]$, $x\in X$, then we obtain two fuzzy sets A^- and A^+ on X, and A^- and A^+ are called lower fuzzy set and upper fuzzy set of A , respectively . Sometime , A is written to be $A=[A^-, A^+]$.

Definition 1.3 Let $A\in[I]^X$, $[a, b]\in[I]$. Then

$$A_{[a,b]} = \{x \in X{:}a \leq A^-(x) \text{ and } b \leq A^+(x)\}$$

is called [a, b]-cut of A(see[3])

It is not difficult to prove that $A_{[a,b]} = A^-_{[a]} \cap A^+_{[b]}$, where $A^-_{[a]} = \{x \in X{:}A^-(x) \geq a\}, A^+_{[b]} = \{x \in X{:}A^+(x) \geq b\}$.

Definition 1.4 A fuzzy set B on R is called bounded closed fuzzy number , if (see [2])

(1) For each $a \in (0,1], B_{[a]} = \{x \in R{:}B(x) \geq a\} \in [R]$;

(2) $KerB = \{x \in R{:}B(x) = 1\} \neq \phi.$

The set of all bounded closed fuzzy numbers on R is denoted by BC(R).

2. Bounded Closed Interval-valued Fuzzy Numbers

Definition 2.1 $A \in [I]^R$ (i.e. A is a interval-valued fuzzy set on R) is called a bounded closed interval-valued fuzzy number (BCIFN, for short), if $A^-, A^+ \in BC(R)$. The set of all BCIFNs on R is denoted by BC[R].

Theorem 2.2 Let $A\in BC[R]$, then for each $[a, b]\in[I]$ $(a\neq 0)$ we have $A_{[a,b]} \in [R]$ and $KerA^- \neq \phi$.

Proof From $A \in BC[R]$ we see that $A^-, A^+ \in BC(R)$, and then $KerA^- \neq \phi$. Take $X_O \in KerA^-$, then for each $[a, b] \in [I]$ $(a \neq 0)$ we have $X_O \in A^-_{[a]}$. It follows from $A^- \leq A^+$ that $X_O \in KerA^+$, and so $X_O \in A^+_{[b]}$. Hence $A^-_{[a]} \cap A^+_{[b]} \neq \phi$. Note that $A^-_{[a]}, A^+_{[b]} \in [R]$, we conclude that $A^-_{[a]} \cap A^+_{[b]} = A_{[a,b]} \in [R]$. □

Remark 2.3 The converse of theorem 2.2 is false as shown by the following example, this is different from bounded closed fuzzy numbers (see theorem 3.11 in [1]).

Example 2.4 Let $A = [A^-, A^+] \in [I]^R$ as follow:

$$A^-(x) = \begin{cases} 0, & x \in (-\infty, 2] \cup [4, +\infty) \\ x-2, & x \in [2,3] \\ 4-x, & x \in [3,4] \end{cases},$$

$$A^+(x) = \begin{cases} 0, & x \in (-\infty, 0) \cup (4, +\infty) \\ 1, & x \in [2,4] \\ 1-0.5x, & x \in [0,1] \\ 0.5x, & x \in [1,2] \end{cases}$$

For each $[a,b] \in [I]$ with $a \neq 0$, clearly, $A^-_{[a]} \subset (2,4)$. Hence $A_{[a,b]} = A^-_{[a]} \cap A^+_{[b]} \in [R]$, and $KerA^- \neq \phi$. But it is clear that $A^+ \notin BC(R)$. Therefore $A \notin BC[R]$. □

Definition 2.5 Let $A \in BC[R]$.

(1) A is called a positive BCIFN, if $A^+(x) = 0$ for each $X \leq 0$. All positive BCIFNs is denoted by $BC[R^+]$

(2) A is called a negative BCIFN, if $A^+(x) = 0$ for each $X \geq 0$. All negative BCIFNs is denoted by $BC[R^-]$.

3. Algebraic Operations on BCIFNs

Definition 3.1 Let $* \in \{+, -, \times, \div\}$ be the binary operation on R. For $A, B \in BC[R]$, $A * B$ is defined as follows:

$$(A*B)(z)=\bigvee_{z=x*y}(A(x)\wedge B(y)) \quad , \quad z\in R \ ,$$

where $A(x)\wedge B(y)=[A^-(x),A^+(x)]\wedge[B^-(y),B^+(y)]$.

Theorem 3.2 Let A, B∈BC[R], then

(1) A∗B ∈BC[R] for $*\in\{+,-,\times\}$;

(2) A÷B ∈BC[R] for $B\in BC[R^+]$ or $B\in BC[R^-]$.

Proof (1) For $*\in\{+,-,\times\}$ and each z∈R,

$$(A*B)(z)=\bigvee_{z=x*y}([A^-(x),A^+(x)]\wedge[B^-(y),B^+(y)])$$

$$=\bigvee_{z=x*y}[A^-(x)\wedge B^-(y),A^+(x)\wedge B^+(y)]$$

$$=\Big[\bigvee_{z=x*y}(A^-(x)\wedge B^-(y)),\bigvee_{z=x*y}(A^+(x)\wedge B^+(y))\Big]$$

$$=[(A^-*B^-)(z),(A^+*B^+)(z)] \text{ (from formula 4.21 in [2])}$$

On the other hand, $(A*B)(z)=[(A*B)^-(z),(A*B)^+(z)]$,
Hence

$$(A*B)^-=A^-*B^- \ , \ (A*B)^+=A^+*B^+. \qquad (3.1)$$

It follows from theorem 4.2.6 in [2] that $A^-*B^-,A^+*B^+\in BC(R)$. This shows $A*B\in BC[R]$.

(2) It is similar to (1). □

Theorem 3.3 Let A, B, C ∈BC[R] , then

(1) A+B=B+A ;

(2) A×B=B×A ;

(3) (A+B)+C=A+(B+C) ;

(4) (A×B)×C=A×(B×C) .

Proof The proofs of (2), (3) and (4) are similar to (1), and hence we only prove (1).

From formula (3.1) and theorem 4.2.7 in [2] we have

$$\begin{aligned} A+B &= [(A+B)^-,(A+B)^+]=[A^-+B^-,A^++B^+] \\ &= [B^-+A^-,B^++A^+]=[(B+A)^-,(B+A)^+] \\ &= B+A\ . \qquad \square \end{aligned}$$

In general, $A\times(B+C)\neq(A\times B)+(A\times C)$. However we have

Theorem 3.4 If $A,B,C\in BC[R^+]$, then

$$A\times(B+C)=(A\times B)+(A\times C)\ .$$

Proof It is similar to theorem 3.3.

Definition 3.5 Let $A,B\in BC[R]$. Define $A\vee B$ and $A\wedge B$ as follows:

$$(A\vee B)(z)=\bigvee_{z=x\vee y}(A(x)\wedge B(y))\ ,\quad z\in R$$

$$(A\wedge B)(z)=\bigvee_{z=x\wedge y}(A(x)\wedge B(y))\ ,\quad z\in R$$

Theorem 3.6 If $A,B\in BC[R]$, then $A\vee B,A\wedge B\in BC[R]$.

Proof Only prove $A\vee B\in BC[R]$. For $z\in R$,

$$\begin{aligned}(A\vee B)(z)&=\bigvee_{z=x\vee y}(A(x)\wedge B(y)) \\ &=\bigvee_{z=x\vee y}([A^-(x),A^+(x)]\wedge[B^-(y),B^+(y)]) \\ &=\bigvee_{z=x\vee y}[A^-(x)\wedge B^-(y),A^+(x)\wedge B^+(y)]\end{aligned}$$

$$= \left[\bigvee_{z=x\vee y} (A^-(x)\wedge B^-(y)), \bigvee_{z=x\vee y} (A^+(x)\wedge B^+(y)) \right]$$

$$= [(A^- \vee B^-)(z),(A^+ \vee B^+)(z)] \ ,$$

(by formula 4.2.10 in [2]).

Note that $(A\vee B)(z) = [(A\vee B)^-(z),(A\vee B)^+(z)]$, so we have

$$(A\vee B)^- = A^- \vee B^-,(A\vee B)^+ = A^+ \vee B^+ \qquad (3.2)$$

Since $A^-,B^-,A^+,B^+ \in BC(R)$, it follows from formula 4.2.12 in [2] that $A^- \vee B^-$, $A^+ \vee B^+ \in BC(R)$, and so $(A\vee B)^-,(A\vee B)^+ \in BC(R)$. Hence $A\vee B \in BC[R]$. □

Theorem 3.7 If A, B, C∈BC[R] , then

(1) $A+(B\vee C) = (A+B)\vee(A+C)$;

(2) $A+(B\wedge C) = (A+B)\wedge(A+C)$;

(3) $A-(B\vee C) = (A-B)\wedge(A-C)$;

(4) $A-(B\wedge C) = (A-B)\vee(A-C)$.

Proof Take (3) for example .

$$A-(B\vee C) = [(A-(B\vee C))^-,(A-(B\vee C))^+]$$
$$=[A^- -(B\vee C)^-, A^+ -(B\vee C)^+] \qquad \text{(from (3.1))}$$
$$=[A^- -(B^- \vee C^-), A^+ -(B^+ \vee C^+)] \qquad \text{(from (3.2))}$$
$$=[(A^- - B^-)\wedge(A^- - C^-),(A^+ - B^+)\wedge(A^+ - C^+)]$$

(by theorem 4.2.10 in [2])

$$=[(A-B)^- \wedge (A-C)^-,(A-B)^+ \wedge (A-C)^+]$$
$$=[((A-B)\wedge(A-C))^-,((A-B)\wedge(A-C))^+]$$
$$=(A-B)\wedge(A-C) \ .$$

□

Similar to theorem 3.7 , we have

Theorem 3.8 If $A,B,C \in BC[R^+]$, then

(1) $A\times(B\vee C) = (A\times B)\vee(A\times C)$;

(2) $A\times(B\wedge C) = (A\times B)\wedge(A\times C)$;

(3) $A\div(B\vee C) = (A\div B)\wedge(A\div C)$;

(4) $A \div (B \wedge C) = (A \div B) \vee (A \div C)$.

4. The Measure for Uncertainty of BCIFNs and Induced Fuzzy Numbers

Given $A \in BC[R]$, for $x \in R, A(x) = [A^-(x), A^+(x)]$. From the view of information theory, if $A^-(x) = A^+(x)$, then information of element x itself is complete . If the information for x is not complete , for example, we only knew roughly the age of Zhang is about 30~35, then membership grade of Zhang is "Young" is a interval number , say , [0. 7, 0. 8] . In this case, $A^-(Zhang) = 0.7, A^+(Zhang) = 0.8$. This shows that the interval number $[A^-(x), A^+(x)]$ is, in reality, a reflection for a mount of information of x , the width of the interval $d(x) = A^+(x) - A^-(x)$ is a quantification of uncertainty for x . Therefore we present the following concept .

Definition 4.1 Given $A \in BC[R]$. Then

$$D(A) = \int_{-\infty}^{+\infty} (A^+(x) - A^-(x))dx \qquad (4.1)$$

is called degree of uncertainty of A .

Note that $A^-, A^+ \in BC(R)$, so infinite integral in (4. 1) is convergent.

Definition 4.2 Let $A \in BC[R]$. We define a fuzzy number $B \in BC(R)$ as follows:

$\forall x \in R, B(x) \in [A^-(x), A^+(x)]$, and

$$B(x) = \begin{cases} 1, & x \in KerA = [m,n] \neq \phi \\ L(x), & x < m \\ R(x), & x > n \end{cases}$$

where , L(x) and R(x) are , respectively , increasing right-continuous function and decreasing left-continuous function , and $\lim_{x \to -\infty} L(x) = 0 = \lim_{x \to +\infty} R(x)$

B is called a fuzzy number induced by A . All fuzzy numbers induced by A is called fuzzy number family induced by A , and written I(A) .

It is not difficult to prove the following two theorem :

Theorem 4.3 Let $A \in BC[R]$, then $(I(A), \le, \vee, \wedge)$ is a complete lattice , and A^- and A^+ are , respectively , its minimal element and maximum element .

Theorem 4.4 Given $A \in BC[R]$. $\forall B \in I(A)$, we define $A_B \in BC[R]$ as $A_B = [B, A^+]$, then

(1) $D(A_B) \le D(A)$;

(2) $\forall B, C \in I(A), B \le C \Rightarrow D(A_C) \le D(A_B)$.

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