INTERVAL—VALUED FUZZY EVENTS AND THEIR PROBABILITY*

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Abstract The concept of interval—valued fuzzy events is introduced,and the probability for this events is defined. The basic properties of the probability are discussed,and the relation between the probability and the probability for fuzzy events is pointed out.

Key words:Interval—valued fuzzy event,Probability,Induced fuzzy event

1. Introduction

The prerequisite discussing probability of fuzzy events[4]is to build membership functions of fuzzy events which conform to the actual situation,but it often is not easy to determine the membership functions,especially,when information is not complete,even unable to set up them. For example,We want to set up the fuzzy event"Bejing is fairly cold on December 30,1998",then we must know,in an all—round way,that some information for Beijing,such as air temperature of all previous years in the city and,cold resistance and living conditions of Beijingese . But it is not easy to obtain all these information, sometimes even is impossible. On the other hand, the standard for"fairly cold"varies from person to person,and from city to city. Hence,we will use interval—valued membership function to replace membership function. On the one hand,this is easy to put into effect and,on the

* Project Supported by Natural Science Foundation of Shandong Province of China

other hand, interval—valued membership function reflect the innate character of fuzzy event. For example, for -20℃, membership degree which belongs to "fairly cold" is better to take [0.7, 0.8] than take 0.8. this is just the basic starting point fo present interval—valued fuzzy events.

2. Preliminaries

In this paper, I denotes real unit interval [0,1], and $[I]=\{\bar{a}=[a^-,a^+]: a^-\leqslant a^+, a^-, a^+\in I\}$. Note that for each $a\in I, a=[a,a]$, and so $I\subset[I]$, $\forall\ \bar{a}=[a^-,a^+], \bar{b}=[b^-,b^+]\in[I]$,

$\bar{a}\wedge\bar{b}=[a^-\wedge b^-, a^+\wedge b^+]$, $\quad \bar{a}\vee\bar{b}=[a^-\vee b^-, a^+\vee b^+]$,

$\bar{a}'=[1-a^+, 1-a^-]$, $\quad \bar{a}\leqslant\bar{b}$ iff $a^-\leqslant b^-$ and $a^+\leqslant b^+$,

$\bar{a}'<\bar{b}$ iff $a^+\leqslant b^-$ $\quad \bar{a}=\bar{b}$ iff $a^-=b^-$ and $a^+=b^+$.

Given a nonempty crisp set X, the mapptng $A: X\to[\mathrm{I}]$ is called and interval—valued fuzzy set on X. All interval—valued fuzzy sets on X is denoted by $[I]^X$.

For $A\in[I]^X, x\in X$, assume that $A(x)=[A^-(x), A^+(x)]$, then we have two fuzzy sets:

$$A^-: X\to I \qquad A^+: X\to I$$
$$x\mapsto A^-(x) \qquad x\mapsto A^+(x),$$

A^- and A^+ are called lower—fuzzy set of A and upper—fuzzy set of A, respectively.

On $[I]^X$, $\vee$, $\wedge$, $'$, $\leqslant$, $=$ are defined in order as follows:

$$\forall\ A, B\in[I]^X, \forall\ x\in X$$

$(A\vee B)(x)=A(x)\vee B(x)=[A^-(x)\vee B^-(x), A^+(x)\ \vee B^+(x)]$,

$(A\wedge B)(x)=A(x)\wedge B(x)=[A^-(x)\wedge B^-(x), A^+(x)\wedge B^+(x)]$,

$A'(x)=(A(x))'=[1-A^+(x), 1-A^-(x)]$,

$A\leqslant B$ iff $A(x)\leqslant B(x)$; $A<B$ iff $A(x)<B(x)$, $A=B$ iff $A(x)=B(x)$.

It is easy to prove the following theorem:

Theorem 2.1 $([I]^X, \vee, \wedge, ', \bar{0}, \bar{1})$ is a complete De Morgan algebra, where, $\bar{0}=[0,0]$ and $\bar{1}=[1,1]$.

3. Interval—valued Fuzzy Events

Definition 3.1 Let Ω be a sampel space, $[I]^\Omega$ be all interval—valued fuzzy sets on Ω, $\tau \subset [I]^{\Omega}$ is called interval—valued fuzzy σ—field (IFσ—field, for short), if

(1) $\bar{1} \in \tau$; (2) if $A \in \tau$ then $A' \in \tau$; (3) if $A_n \in \tau, n=1,2,\cdots$, then $\bigvee_{n=1}^{\infty} A_n \in \tau$.

The member of τ is called an interval—valued fuzzy event, simply, IF—event. $\bar{1}$ is called IF—certain event, and $\bar{0}$ IF—impossible event.

Example 3.2 All constant interval—valued fuzzy sets on Ω forms an IFσ—field, where, by constant interval—valued fuzzy set A on Ω is meant $A(x) \equiv [a,b] \in [I], \forall\ x \in \Omega$.

For $A, B \in \tau \subset [I]^{\Omega}$, according to the operations of interval—valued fuzzy sets, we obtain some operations for IF—events as follows:

$$A \vee B, A \wedge B, A', A \leqslant B, A=B.$$

In addition, we also introduce two operations as follows:

for $A, B \in \tau$, and $B < A, x \in \Omega$,

$(A-B)(X) = A(X) - B(X) = [A^-(x) - B^+(x), A^+(x) - B^-(x)]$,

$A^T(x) = [A^+(x), A^-(x)]$.

Note that for IF—events A, generally

$$A \vee A' \neq \bar{1}, A \wedge A' \neq \bar{0}.$$

This is different from crisp events.

It is easy to prove the following theorem:

Theorem 3.3 Let $A, B, C \in \tau \subset [I]^{\Omega}$, then

(1) $A \vee B = B \vee A$; $A \wedge B = B \wedge A$.

(2) $(A \vee B) \vee C = A \vee (B \vee C)$; $(A \wedge B) \wedge C = A \wedge (B \wedge C)$.

(3) $(A \vee B) \wedge C = (A \wedge C) \vee (B \wedge C)$; $(A \wedge B) \vee C = (A \vee C) \wedge (B \vee C)$.

(4) $(A \vee B)' = A' \wedge B'$; $(A \wedge B)' = A' \vee B'$.

Definition 3.4 Let $A, B \in \tau \subset [I]^{\Omega}$. If $A \wedge B = \bar{0}$, then A and B are called exclusive, and in the case, $A \vee B$ is written as $A+B$.

4. Probability of Interval—valued Fuzzy Events

Definition 4.1 Let $\Omega = \{\omega_1, \omega_2, \cdots\}$ be a sample space, and probability of ω_i

be $p(\omega_i\}$, $i=1,2,\cdots$, and $\sum_{i=1}^{\infty} p(\omega_i)=1$, and $A\in\tau\subset[I]^{\Omega}$. Then probability of A is defined by

$$P(A)=\sum_{i=1}^{\infty} A(\omega_i)p(\omega_i)=[\sum_{i=1}^{\infty} A^{-}(\omega_i)p(\omega_i),\sum_{i=1}^{\infty} A^{+}(\omega_i)p(\omega_i)].$$

Remark 4.2 For $A\in\tau\subset[I]^{\Omega}$, supose that $A=[A^{-},A^{+}]$, then A^{-} and A^{+} are called lower—fuzzy event of A and upper—fuzzy event of A, respectively. From definition of probabilify for fuzzy event (ref. [1]) we see that $P(A)=[P(A^{-}),P(A^{+})]$, that is, $P(A)$ is an interval number, its left hand point and right hand point are, respectively, the probability of lower—fuzzy A^{-} of A and that of upper—fuzzy event A^{+} of A.

Example 4.3 For shooting practice, $\Omega=\{0,1,2,\cdots,10\}$. The interval—valued fuzzy event "hit the high—point rings" is defined as follows:

$$A=[0.6,0.8]/6+[0.7,0.9]/7+[0.8,1]/8+[o.9,1]/9+[1,1]/10.$$

If assume the probability hitting each point ring is same, i.e., $p(k)=\frac{1}{11}$, $k=0,1,\cdots,10$, then

$$P(A)=([0.6,0.8]+[0.7,0.9]+[0.8,1]+[0.9,1]+[1,1])\times\frac{1}{11}$$
$$=[0.364,0.427].$$

We now discuss the basic properties of probability for IF—events.

Theorem 4.4 Let $A,B\in\tau\subset[I]^{\Omega}$, then

(1) if $A\leqslant B$, then $P(A)\leqslant P(B)$;

(2) $P(A')=[1,1]-P(A)$.

Proof (1) If is clear.

(2) First, for fuzzy event B on Ω we have $B'=1-B$, i,e., $\forall\ \omega\in\Omega$, $B'(\omega)=1-B(\omega)$. Also, it follows from theorem 2~17 in [3] that the probability of B' is $P(B')=1-P(B)$. On the other hand, for $A\in\tau$, suppose that $A=[A^{-},A^{+}]$, then from theorem 3.1 in [2] we have $(A')^{-}=(A^{+})'$ and $(A')^{+}=(A^{-})'$. By remark 4.2 we get

$$P(A')=[P((A')^{-}),P((A')^{+})]=[P((A^{+})'),P((A^{-})')]$$
$$=[1-P(A^{+}),1-P(A^{-})]=[1,1]-[P(A^{-}),P(A^{+})]$$

$=[1,1]-P(A)$. □

Theorem 4.5 Let $A,B\in\tau\subset[I]^{\Omega}$, then $P(A\vee B)=P(A)+P(B)-P((A\wedge B)^{T})$.

Proof First, it follws from theorem 3.1 in[2]that $(A\vee B)^{-}=A^{-}\vee B^{-}$, $(A\vee B)^{+}=A^{+}\vee B^{+}$. Then from theorem 2~17 in [3] we have

$$
\begin{aligned}
&P(A\vee B)\\
&=[P((A\vee B)^{-}),P((A\vee B)^{+})]\\
&=[P(A^{-}\vee B^{-}),P(A^{+}\vee B^{+})]\\
&=[P(A^{-})+P(B^{-})-P(A^{-}\wedge B^{-}),P(A^{+})+P(B^{+})-P(A^{+}\wedge B^{+})]\\
&=[P(A^{-}),P(A^{+})]+[P(B^{-}),P(B^{+})]-[P(A^{+}\wedge B^{+}),P(A^{-}\wedge B^{-})]\\
&=P(A)+P(B)-[P((A\wedge B)^{+}),P((A\wedge B)^{-})]\\
&=P(A)+P(B)-P((A\wedge B)^{T}).
\end{aligned}
$$
□

Corollary 4.6 Let $A_i\in\tau\subset[I]^{\Omega}$, $i=1,2,\cdots,n$, and A_i and $A_j(i\neq j)$ be exclusive, then

$$P(A_1+A_2+\cdots+A_n)=P(A_1)+P(A_2)+\cdots+P(A_n).$$

Corollary 4.7 Let $A_i\in\tau\subset[I]^{\Omega}$, $i=1,2,\cdots,n$, then

$$P(A_1\vee A_2\vee\cdots\vee A_n)\leqslant P(A_1)+P(A_2)+\cdots+P(A_n).$$

Theorem 4.8 Let $A,B\in\tau\subset[I]^{\Omega}$, and $B<A$, then

$$P(A-B)=P(A)-P(B).$$

Proof From the definition of $A-B$ we have

$$(A-B)^{-}=A^{-}-B^{+},(A-B)^{+}=A^{+}-B^{-},$$

and then

$$
\begin{aligned}
P(A-B)&=[P((A-B)^{-}),P((A-B)^{+})]\\
&=[P(A^{-}-B^{+}),P(A^{+}-B^{-})]\\
&=[P(A^{-})-P(B^{+}),P(A^{+})-P(B^{-})]\\
&=[P(A^{-}),P(A^{+})]-[P(B^{-}),P(B^{+})]=P(A)-P(B).
\end{aligned}
$$
□

5. Induced Fuzzy Events and Their Probability

Theorem 5.1 Let $\tau\subset[I]^{\Omega}$ be an IFσ-field on Ω. $\forall\ A\in\tau$, $\forall\ \omega\in\Omega$, take

$N_A^{(\omega)} \in [A^-(\omega), A^+(\omega)]$, then we obtain a fuzzy set N_A on Ω. Put $N_\tau = \{N_A: A \in \tau\}$, then

(1) $1 \in N_\tau$, where , $1(\omega) \equiv 1, \forall\ \omega \in \Omega$;

(2) if $B \in N_\tau$, then, $B' \in N_\tau$;

(3) if $B_n \in N_\tau, n=1,2,\cdots$, then $\bigvee_{n=1}^{\infty} B_n \in N_\tau$.

Proof (1) It follows from $\bar{1} \in \tau$ that $1 \in N_\tau$.

(2) From $B \in N_\tau$, there is $A \in \tau$ such that $A^-(\omega) \leqslant B(\omega) \leqslant A^+(\omega)$ for each $\omega \in \Omega$, and then $1 - A^+(\omega) \leqslant 1 - B(\omega) \leqslant 1 - A^-(\omega)$, that is, $(A^+)'(\omega) \leqslant B'(\omega) \leqslant (A^-)'(\omega)$. By theorem 3. 1 in [2], $(A')^-(w) \leqslant B'(w) \leqslant (A')^+(\omega)$. Since τ is IFσ—field, we see that $A' \in \tau$. This shows that $B' \in N_\tau$.

(3) Since $B_n \in N_\tau$, there exists $A_n \in \tau$, such that $A_n^-(\omega) \leqslant B_n(\omega) \leqslant A_n^+(\omega)$ for each $\omega \in \Omega, n=1,2,\cdots$, and then $\bigvee_{n=1}^{\infty} A_n^-(\omega) \leqslant \bigvee_{n=1}^{\infty} B_n(\omega) \leqslant \bigvee_{n=1}^{\infty} A_n^+(w)$, that is, $(\bigvee_{n=1}^{\infty} A_n)^-(\omega) \leqslant (\bigvee_{n=1}^{\infty} B_n)(w) \leqslant (\bigvee_{n=1}^{\infty} A_n)^+(\omega)$. It follows from τ is an IFσ—field that $\bigvee_{n=1}^{\infty} A_n \in \tau$, This shows that $\bigvee_{n=1}^{\infty} B_n \in N_\tau$. □

Definition 5. 2 Let $\tau \subset [I]^\Omega$. N_τ in theoren 5. 1 is called a fuzzy σ—field induced by τ. For $A \in \tau$, N_A is called a fuzzy event induced by A. All fuzzy events induced by A is denoted by IN(A).

It is easy to prove the following theorem:

Theorem 5. 3 Let $A \in \tau \subset [I]^\Omega$, then (IN$(A)$, $\vee$, $\wedge$,$\leqslant$,$'$)forms a complete lattice, and A^+ and A^- are, respectively, its greatest and least lelments.

Theorem 5. 4 Let $A \in \tau \subset [I]^\Omega$, and probability of A is $P(A) = [P(A^-), P(A^+)]$. Then for each $\xi \in [P(A^-), P(A^+)]$. there is $N_A \in$ IN(A) such that the probability of fuzzy event N_A is $P(N_A) = \xi$.

Proof Clearly, there is $\alpha \in [0,1]$such that $\xi = (1-\alpha)P(A^-) + \alpha P(A^+)$, Define fuzzy set N_A on Ω as follows:

$$N_A(\omega) = (1-\alpha)A^-(\omega) + \alpha A^+(\omega), \forall\ \omega \in \Omega \qquad (*)$$

Since the derivative of function $f(x) = A^-(\omega)(1-x) + A^+(\omega)(x)$ $(x \in [0,$

1])is $f'(x)=A^+(\omega)-A^-(\omega)\geqslant 0$, $f(x)$ is increasing function, and so $A^-(\omega)\leqslant N_A(\omega)\leqslant A^+(\omega)$. Hence $N_A\in \mathrm{IN}(A)$. From formula (3. 92)and (3. 93)in [1]we get probability of N_A:

$$P(N_A)=\sum_{i=1}^{\infty} N_A(\omega_i)p(\omega_i)=\sum_{i=1}^{\infty}((1-\alpha)A^-(\omega_i)p(\omega_i)+\alpha A^+(\omega_i)p(\omega_i))$$
$$=(1-\alpha)\sum_{i=1}^{\infty}A^-(\omega_i)p(\omega_i)+\alpha\sum_{i=1}^{\infty}A^+(\omega_i)p(\omega_i)$$
$$=(1-\alpha)P(A^-)+\alpha P(A^+)=\xi.$$

References

[1] Wang Peizhuang and Han Liyan, Applied Fuzzy Mathematics, Beijing Economic College Press, 1989, Beijing, China (in Chinese).

[2] Meng Guangwu ,Basic theory for interval—valued fuzzy sets, Applied Mathematics, 2(1993), 212～217(in Chinese).

[3] Wang Shitong etal. Applications of Fuzzy Mathematics to Artificial Intelligence, Engineering Industry Press, 1991, Beijing, China (in Chinse).

[4] L. A. Zadeh, Probabilify measures of fuzzy events, J. Math. Anal. Appl. 23(1968), 421～427.