Convergence theorems of the Choquet integral

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1. Introduction

Since Sugeno[8] introduced the concept of fuzzy measures in 1974, The fuzzy integral with respect to a fuzzy measure is generally developed, such as Sugeno[8], Ralescu[7], Zhao[13], Suarez[9], Wu[11], etc. The common property of all the various kinds of fuzzy integrals is that they are monotonic functional from the space of nonnegative measurable functions to $[0,\infty]$.

Just based on the property, Murofushi and Sugeno also view the Choquet integral which is introduced by Choquet[1] in 1956, as a kind of fuzzy integrals, and have given a much extensive study in[3],[4],[5],[6].

It is well-known that convergence theorems are very important in classical integral theory. But for the Choquet integral, convergence theory is not enough. Since so far, only the monotone convergence theorem is shown(see[4]). Whether can other convergence theorems be established? The answer is just the paper's purpose.

In the paper, we will show various kinds of generalized convergence theorems of the Choquet integral, these include generalized monotone convergence theorem, generalized Fatou's lemmas, etc.

The rest of the paper is divided into two parts. Section 2 will give some concepts and reviews on fuzzy measures and the Choquet integral as preparation. Section 3 will show the main results of this paper.

2. Fuzzy measures and the Choquet integral.

Let X be a nonempty classical set, $\mathscr{A}$ a σ-algebra formed by the subsets of X, $(X, \mathscr{A})$ the measurable space.

Definition 2.1[4]. Let $\mu: \mathscr{A} \to [0,\infty]$ be a set-function. Then

(1) μ is said to be empty-null if $\mu(\Phi) = 0$;

(2) μ is said to be monotone if $\mu(A) \leqslant \mu(B)$ whenever $A \subset B$;

(3) μ is said to be continuous from below if $\mu(A_n) \uparrow \mu(A)$ whenever $A_n \uparrow A$.

(4) μ is said to be conditionally continuous from above if $\mu(A_n) \downarrow \mu(A)$ whenever $A_n \downarrow A$, and $\mu(A_n) < \infty$ for some $n \geqslant 1$.

Definition 2.2[8]. A set-function $\mu: \mathscr{A} \to [0,\infty]$ is called a fuzzy measure if it is empty-null and monotone.

The triplet $(X, \mathscr{A}, \mu)$ is called a fuzzy measure space.

Let the symbol M(X) denote the set of all fuzzy measures on $(X, \mathscr{A})$.

Proposition 2.1[12]. Let $\{\mu_n\}$ be a sequence of fuzzy measures on $(X, \mathscr{A})$ i. e. $\{\mu_n\} \subset M(X)$, and μ a set-function from $\mathscr{A}$ to $[0,\infty]$. If $\{\mu_n\}$ converges to μ (setwise convergence, i. e $\mu_n(A) \to \mu(A)$ for every $A \in \mathscr{A}$ simply write as $\mu_n \to \mu$), then

(1) μ is a fuzzy measure;

(2) μ is continous from below whenever $\mu_n (n \geqslant 1)$ is continuous from below.

Proposition 2.2[12]. Let $\{\mu_n\}$ be a sequence of fuzzy measures on $(X, \mathscr{A})$, and μ a set function from $\mathscr{A}$ to $[0,\infty]$. If $\{\mu_n\}$ converges uniformly to μ, then $\mu_n (n \geqslant 1)$ is conditionally continuous from above implies that μ is conditionally continuous from above.

All the $\mathscr{A}$-measurable functions from $\mathscr{A}$ to $[0,\infty]$ is denoted by F(X).

Definition2.3[1]. Let $f \in F(X)$, $\mu \in M(X)$. Then the Choquet integral of f with respect to μ, which denoted by $(C)\int f d\mu$ is as follow:

$$(C)\int f d\mu = \int_0^\infty \mu((f > t)) dt$$

Where the right side integral is Lebesgue integral, and $(f > t)$ stand for $\{x \in X : f(x) > t\}$.

The next proposition follows immediately from the definition. Statements (1), (2), (3) were shown in [1], and (4), (5), (6) were shown in [4].

Proposition 2.3. Let $\{f_n\} \subset F(X)$, $f, g \in F(X)$.

(1) $f \leqslant g$ implies $(C)\int f d\mu \leqslant (C)\int g d\mu$

(2) $(C)\int a \cdot f d\mu = a (C)\int f d\mu, a \geqslant 0$

(3) If m is a σ — additive, i. e. m is a classical measure, then the Choquet integral coincides with the Lebesgue integral,

$$(C)\int fdm = \int fdm$$

(4) $(C)\int I_A d\mu = \mu(A)$, $A \in \mathscr{A}$, I_A denotes the characteristic function of A.

(5) Let μ be continuous from below. If $f_n \uparrow f$, then

$$(C)\int f_n d\mu \uparrow (C)\int fd\mu.$$

(6) Let μ be conditionally continuous from above. If $f_n \downarrow f$, and if $f_1 \leqslant g$ for some $g \in F(X)$ which $(C)\int gd\mu < \infty$, then

$$(C)\int f_n d\mu \downarrow (C)\int fd\mu$$

Definition 2. 4[4]. A set $N \in \mathscr{A}$ is called a null-set (with respect to μ iff $\mu(A \cup N) = \mu(A)$, for all $A \in \mathscr{A}$.

By using the "null set", the "almost everywhere" concept is defined as: "P(X) a. e" means that there is a null set N, such that P(x) is true for all $x \in N^c$, where P(x) is a proposition concerning the points of X.

Proposition 2. 4[4]. If W^c is a null set, then for every measurable function f,

$$(C)\int fd\mu = (C)\int f_w d\mu_w$$

where f_w is the restriction of f on W, μ_w is similar.

Proposition 2. 5[4] Given a measurable set N, the following conditions are equivalent,

(1) N is a null set

(2) $(C)\int fd\mu = (C)\int gd\mu$ for all $f, g \in F(X)$, s. t. $f(x) = g(x), \forall x \in N^c$.

3. Convergence theorems.

Theorem 3. 1. Let $\{f_n\} \subset F(X)$, $f \in F(X)$.

(1) Let μ be continuous from below. If $f_n \uparrow f$ a. e, then

$$(C)\int f_n d\mu \uparrow (C)\int fd\mu.$$

(2) Let μ be conditionally continuous from above. If $f_n \downarrow f$ a. e, and if $f_1 \leqslant g$ a. e for some $g \in F(X)$ which $(C)\int g d\mu < \infty$, then

$$(C)\int f_n d\mu \downarrow (C)\int f d\mu.$$

Lemma 3. 1. Let $\{\mu_n\} \subset M(X), \mu \in M(X), f \in F(X)$.

(1) If $\mu_n \uparrow \mu$, then

$$(C)\int f d\mu_n \uparrow (C)\int f d\mu.$$

(2) If $\mu_n \downarrow \mu$, and

$$(C)\int f d\mu_m < +\infty \text{ for some } m \geqslant 1,$$

then

$$(C)\int f d\mu_n \downarrow (C)\int f d\mu.$$

Theorem 3. 2. Let $\{f_n\} \subset F(X)$, $f \in F(X)$, $\{\mu_n\} \subset M(X), \mu \in M(X)$. If $f_n \uparrow f$ a. e, $\mu_n \uparrow \mu$, then μ is continuous from below implies

$$(C)\int f_n d\mu_n \uparrow (C)\int f d\mu.$$

Theorem 3. 3. Let $\{f_n\} \subset F(X)$, $f \in F(X)$, $\{\mu_n\} \subset M(X), \mu \in M(X)$, and $(C)\int f_n d\mu_n < +\infty$ for some $n \geqslant 1$, If $f_n \downarrow f$ a. e, $\mu_n \downarrow \mu$, then μ is conditionally continuous from above implies that

$$(C)\int f_n dd\mu_n \downarrow (C)\int f d\mu.$$

Theorem 3. 4. Let $\{f_n\} \subset F(X)$, $\{\mu_n\} \subset M(X)$. If $\underline{\lim}_{n\to\infty} \mu_n$ is continuous from below, then

$$(C)\int \underline{\lim}_{n\to\infty} f_n d(\underline{\lim}_{n\to\infty} \mu_n) \leqslant \underline{\lim}_{n\to\infty} (C)\int f_n d\mu_n$$

Theorem 3. 5. Let $\{f_n\} \subset F(X)$, $\{\mu_n\} \subset M(X)$, and $(C)\int (\sup_{k\geqslant n} f_k) d(\sup_{k\geqslant n} \mu_k) < \infty$ and for some $n \geqslant 1$. If $\overline{\lim}_{n\to\infty} \mu_n$ is conditionally continuous from above, then

$$\overline{\lim}_{n\to\infty} (C)\int f_n d\mu_n \leqslant (C)\int (\overline{\lim}_{n\to\infty} f_n) d(\overline{\lim}_{n\to\infty} \mu_n)$$

Therem3. 6. Let $\{f_n\} \subset F(X)$, $f \in F(X)$, $\{\mu_n\} \in M(X)$, $\mu \in M(X)$, and μ is both continuous from below and conditionally continuous from above. If (C)

$\int (\sup_{k\geqslant n} f_k)\, d(\sup_{k\geqslant n} \mu_k) < \infty$ for some $n \geqslant 1$, then $f_n \to f$ a. e, $\mu_n \to \mu$ implies

$$(C)\int f_n d\mu_n \to (C)\int f d\mu.$$

Corollary 3. 4. Let $\{f_n\} \subset F(X)$, $\mu \in M(X)$.

(1) If μ is continuous from below, then

$$(C)\int \underline{\lim}_{n\to\infty} f_n d\mu \leqslant \underline{\lim}_{n\to\infty} (C)\int f_n d\mu.$$

(2) If μ is conditionally continuous from above, $\int \sup_{k\geqslant n} f_k d\mu < \infty$ for some $n \geqslant 1$, then

$$\overline{\lim}_{n\to\infty} (C)\int f_n d\mu \leqslant (C)\int \overline{\lim}_{n\to\infty} f_n d\mu.$$

If we set $f_n = f (n \geqslant 1)$, then we have

Corollary 3. 2. Let $f \in F(X)$, $\{\mu_n\} \subset M(X)$. Then

(1) $(C)\int fd(\underline{\lim}_{n\to\infty}\mu_n) \leqslant \underline{\lim}_{n\to\infty} (C)\int f d\mu_n$

(2) If $(C)\int fd(\sup_{k\geqslant n} \mu_k < \infty$ for some $n \geqslant 1$, then

$$\overline{\lim}_{n\to\infty} (C)\int f d\mu_n \leqslant (C)\int fd(\overline{\lim}_{n\to\infty}\mu_n)$$

Definition 3. 1. Let $\{f_n\} \subset F(X)$, $f \in F(X)$, $\mu \in M(X)$. If $(C)\int |f_n - f| d\mu \to 0 (n \to \infty)$, then we say that $\{f_n\}$ is C-mean convergent to f. It is simply written by $f_n \to f$(c-m).

Theorem 3. 6. Let $\{f_n\} \subset F(X)$, $f \in F(X)$. If $f_n \to f$(c-m) then $f_n \xrightarrow{\mu} f$.

Theorem 3. 7. Let $\{f_n\} \subset F(X)$, $f \in F(X)$, $\mu \in M(X)$, $\mu(X) < \infty$. If there exists a $g \in F(X)$, $(C)\int g d\mu < \infty$, such that $|f_n - f| \leqslant g$ for each $n \geqslant 1$, then $f_n \xrightarrow{\mu} f$ implies $f_n \to f(c - m)$.

Corollary 4. 1. If the conditions in theorem 3. 7 are satisfied, then $f_n \longrightarrow f$(c-m) is equivalent to $f_n \xrightarrow{\mu} f$.

Theorem 3. 8. Let the conditions in lemma 4. 1 are satisfied, further assume μ is conditionally continuous from above. If there exist some $m \geqslant 1$, such that $(C)\int (\sup_{K\geqslant m} f_k) d\mu < \infty$, and assume $\lim_{n\to\infty} (C)\int f_n d\mu$ exists, then $f_n \xrightarrow{\mu} f$

implies

$$(C)\int f_n d\mu \to (C)\int f d\mu.$$

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