A NEW CLASS OF RANDOM GENERALIZED NONLINEAR QUASI-VARIATIONAL INCLUSIONS FOR RANDOM FUZZY MAPPINGS

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Abstract. The main purpose of this paper is to introduce and study introduce and study a new class of random generalized nonlinear quasivariational inclusions for random fuzzy mappings and construct a new iterative algorithm for this kind of quasivariational inclusions. We also prove the existence of random solutions for this class of random quasivariational inclusions and the convergence of random iterative sequences generated by the algorithm.

Keywords: Quasivariational inclusion; Random fuzzy mapping; Algorithm; Existence; Convergence

1. Introduction

It is well known that variational inequalities, introduced by Hartman and Stampacchia [15] in the early sixties, are a very powerful tool of the current mathematical technology. Variational inequalities have been extended and generalized to study a wide class of problems arising in mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium and engineering sciences etc. Quasivariational inequalities are generalized forms of variational inequalities in which the constraint set depends on the solution. For details we refer to [1, 2, 4, 6, 30, 35].

In 1989, Chang and Zhu [13] were first to introduce and study a class of variational inequalities for fuzzy mappings. Recently, several kinds of variational inequalities and complementarity problems for fuzzy mappings were considered by many authors (see[6, 7, 12, 19, 20, 23, 28, 29, 33, 34]).

In 1991, Chang and Huang [8, 9] were first to introduce and study some new class of complementarity problems and variational inequalities for set-valued mappings with compact values in Hilbert spaces. In the recent paper [16], Hassouni and Moudafi studied a new class of variational inclusions, which included many variational and quasivariational inequalities considered by Noor [31, 32], Isac [27], Siddiqi and Ansari [36, 37] as special cases.

On the other hand, the random variational inequality and random quasi-variational inequality problems have been introduced and studied by Chang [6], Chang and Huang [10, 11], Chang and Zhu [14], Huang [24, 25], Husain, Tarafdar and Yuan [26], Tan, Tarafdar and Yuan [39], Tan [38] and Yuan [40].

Recently, the author [21, 22] were first to introduce and study the random complementarity problems, random quasivariational inequalities and random variational inclusions for random fuzzy mappings.

In this paper, we introduce and study a new class of random generalized nonlinear quasivariational inclusions for random fuzzy mappings and construct a new iterative algorithm for this kind of quasivariational inclusions. We also prove the existence of random solutions for this class of random quasivariational inclusions and the convergence of random iterative sequences generated by the algorithm.

2. Preliminaries

Throughout this paper, let $(\Omega, \mathcal{A})$ be a complete σ-finite measure space and H be a separable real Hilbert space endowed with a norm $\|\cdot\|$, and inner product $(\cdot,\cdot)$. We denote by $\mathcal{B}(H)$, 2^H, $CB(H)$ and $H(\cdot,\cdot)$ the class of Borel σ-fields in H, the family of all nonempty subsets of H, the family of all nonempty closed bounded subsets of H and the Hausdorff metric on $CB(H)$, respectively.

Definition 2.1. A mapping $x : \Omega \to H$ is said to be measurable if for any $B \in \mathcal{B}(H)$, $\{t \in \Omega : x(t) \in B\} \in \mathcal{A}$.

Definition 2.2. A mapping $T : \Omega \times H \to H$ is called a random operator if for any $x \in H$, $T(t, x) = x(t)$ is measurable. A random operator T is said to be continuous if for any $t \in \Omega$, the mapping $T(t, \cdot) : H \to H$ is continuous.

Definition 2.3. A set-valued mapping $V : \Omega \to 2^H$ is said to be measurable if for any $B \in \mathcal{B}$, $V^{-1}(B) = \{t \in \Omega : V(t) \cap B \neq \emptyset\} \in \mathcal{A}$.

Definition 2.4. A mapping $u : \Omega \to H$ is called a measurable selection of a set-valued measurable mapping $V : \Omega \to 2^H$ if u is measurable and for any $t \in \Omega$, $u(t) \in V(t)$.

Definition 2.5. A mapping $V : \Omega \times H \to 2^H$ is called a random set-valued mapping if for any $x \in H$, $V(\cdot, x)$ is measurable. A random set-valued mapping $V : \Omega \times H \to CB(H)$ is said to be H-continuous if for any $t \in \Omega$, $V(t, \cdot)$ is continuous in the Hausdorff metric.

Let N be any set and $\mathcal{F}(H)$ be a collection of all fuzzy sets over H. A mapping F from N into $\mathcal{F}(H)$ is called a fuzzy mapping. If F is a fuzzy mapping on H, then for any $x \in N$, $F(x)$(denote it by F_x, in the sequel) is a fuzzy set on H and $F_x(y)$ is the membership function of y in F_x.

Let $M \in \mathcal{F}(H)$, $q \in [0,1]$. Then the set

$$(M)_q = \{x \in H : M(x) \geq q\}$$

is called a q-cut set of M.

Definition 2.6. A fuzzy mapping $F : \Omega \to \mathcal{F}(H)$ is called measurable, if for any given $\alpha \in (0,1]$, $(F(\cdot))_\alpha : \Omega \to 2^H$ is a measurable set-valued mapping.

Definition 2.7. A fuzzy mapping $F : \Omega \times H \to \mathcal{F}(H)$ is called a random fuzzy mapping, if for any given $x \in H$, $F(\cdot, x) : \Omega \to \mathcal{F}(H)$ is a measurable fuzzy mapping.

Obviously, the random fuzzy mapping includes set-valued mapping, random set-valued mapping and fuzzy mapping as the special cases.

Let $T,\ A : \Omega \times H \to \mathcal{F}(H)$ be two random fuzzy mappings satisfying the following condition (I):

(I) There exist two mappings $a,\ b : H \to (0,1]$ such that

$$(T_{t,x})_{a(x)} \in CB(H),\ (A_{t,x})_{b(x)} \in CB(H),\ \forall (t,x) \in \Omega \times H.$$

By using the random fuzzy mappings T and A, we can define two random set-valued mappings $\widetilde{T}$ and $\widetilde{A}$ as follows:

$$\widetilde{T} : \Omega \times H \to CB(H),\ x \longmapsto (T_{t,x})_{a(x)},\ \forall (t,x) \in \Omega \times H,$$

$$\widetilde{A} : \Omega \times H \to CB(H),\ x \longmapsto (A_{t,x})_{b(x)},\ \forall (t,x) \in \Omega \times H.$$

In the sequel, $\widetilde{T}$ and $\widetilde{A}$ are called the random set-valued mappings induced by the random fuzzy mappings T and A, respectively.

Given mappings $a,\ b : H \to (0,1]$, random fuzzy mappings $T,\ A : \Omega \times H \to \mathcal{F}(H)$ and random operators $f,\ p,\ g : \Omega \times H \to H$. Suppose $\varphi : H \times H \to R \cup \{+\infty\}$ such that for each fixed $y \in H$, $\varphi(\cdot, y) : H \to R \cup \{+\infty\}$ is a proper convex lower semicontinuous function on H and $\operatorname{Im} g \bigcap \operatorname{dom}(\partial\varphi(\cdot, y)) \neq \emptyset$ for each $y \in H$, where $\partial\varphi(\cdot, y)$ denotes the subdifferential of function $\varphi(\cdot, y)$. We consider the following problem:

Find measurable mappings $u,\ w,\ y : \Omega \to H$, such that for all $t \in \Omega$, $v \in H$,

$$\left.\begin{array}{r} T_{t,u(t)}(w(t)) \geq a(u(t)),\ A_{t,u(t)}(y(t)) \geq b(u(t)), \\ g(t,u(t)) \bigcap \operatorname{dom}(\partial\varphi(\cdot, u(t))) \neq \emptyset, \\ (f(t,w(t)) - p(t,y(t)), v - g(t,u(t))) \geq \varphi(g(t,u(t)), u(t)) - \varphi(v, u(t)). \end{array}\right\} \tag{2.1}$$

This problem is called a random generalized nonlinear quasivariational inclusion for random fuzzy mappings.

If $\varphi(x,y) = \varphi(x)$ for all $y \in H$, then the problem (2.1) is equivalent to finding measurable mappings $u,\ w,\ y : \Omega \to H$, such that for all $t \in \Omega$, $v \in H$,

$$\left.\begin{array}{r} T_{t,u(t)}(w(t)) \geq a(u(t)),\ A_{t,u(t)}(y(t)) \geq b(u(t)), \\ g(t,u(t)) \bigcap \operatorname{dom}(\partial\varphi) \neq \emptyset, \\ (f(t,w(t)) - p(t,y(t)), v - g(t,u(t))) \geq \varphi(g(t,u(t))) - \varphi(v), \end{array}\right\} \tag{2.2}$$

which is called a random generalized nonlinear variational inclusion for random fuzzy mappings considered by Huang [22].

If T and A are two random set-valued mappings, and $a(x) = b(x) = 1$, $\forall x \in H$, then the problem (2.1) is equivalent to finding measurable mappings $u, w, y : \Omega \to H$, such that for all $t \in \Omega$, $v \in H$,

$$\left.\begin{array}{c} w(t) \in T(t,u(t)),\ y(t) \in A(t,u(t)),\ g(t,u(t)) \bigcap \mathrm{dom}(\partial\varphi(\cdot,u(t))) \neq \emptyset, \\ (f(t,w(t)) - p(t,y(t)), v - g(t,u(t))) \geq \varphi(g(t,u(t)),u(t)) - \varphi(v,u(t)) \end{array}\right\} \quad (2.3)$$

and the problem (2.2) is equivalent to finding measurable mappings $u, w, y : \Omega \to H$, such that for all $t \in \Omega$, $v \in H$,

$$\left.\begin{array}{c} w(t) \in T(t,u(t)),\ y(t) \in A(t,u(t)),\ g(t,u(t)) \bigcap \mathrm{dom}(\partial\varphi) \neq \emptyset, \\ (f(t,w(t)) - p(t,y(t)), v - g(t,u(t))) \geq \varphi(g(t,u(t))) - \varphi(v), \end{array}\right\} \quad (2.4)$$

It is clear that the random generalized nonlinear quasivariational inclusion (2.1) includes many kinds of variational inequalities and quasivariational inequalities of [6, 8-11, 16, 18, 20-22, 24, 25, 27, 31, 32, 34, 36, 37] as special cases.

3. Random Iterative Algorithm

We first give the following lemmas.

Lemma 3.1.(Chang [5]) *Let $V : \Omega \times H \to CB(H)$ be a H-continuous random set-valued mapping. Then for any measurable mapping $u : \Omega \to H$, the set-valued mapping $V(\cdot, u(\cdot)) : \Omega \to CB(H)$ is measurable.*

Lemma 3.2.(Chang [5]) *Let $V, W : \Omega \to CB(H)$ be two measurable set-valued mappings, $\epsilon > 0$ be constant and $u : \Omega \to H$ be a measurable selection of V. Then there exists a measurable selection $v : \Omega \to H$ of W such that for all $t \in \Omega$,*

$$\|u(t) - v(t)\| \leq (1+\epsilon)H(V(t), W(t)).$$

Lemma 3.3. *Measurable $u, w, y : \Omega \to H$ are a solution of problem (2.1) if and only if for all $t \in \Omega$, $w(t) \in \widetilde{T}(t,u(t))$, $y(t) \in \widetilde{A}(t,u(t))$ and*

$$g(t,u(t)) = J_{\alpha(t)}^{\varphi(\cdot,u(t))}(g(t,u(t)) - \alpha(t)(f(t,w(t)) - p(t,y(t)))), \quad (3.1)$$

where $\alpha : \Omega \to (0,\infty)$ is a measurable function and $J_{\alpha(t)}^{\varphi(\cdot,u(t))} = (I + \alpha(t)\partial\varphi(\cdot,u(t)))^{-1}$ is the so-called proximal mapping on H.

Proof. From the definition of $J_{\alpha(t)}^{\varphi(\cdot,u(t))}$ one has

$$g(t,u(t)) - \alpha(t)(f(t,w(t)) - p(t,y(t))) \in g(t,u(t)) + \alpha(t)\partial\varphi(\cdot,u(t))(g(t,u(t))), \quad \forall t \in \Omega,$$

hence

$$p(t,y(t)) - f(t,w(t)) \in \partial\varphi(\cdot,u(t))(g(t,u(t))), \quad \forall t \in \Omega.$$

From definition of $\partial\varphi(\cdot, u(t))$ we have

$$\varphi(v, u(t)) \geq \varphi(g(t, u(t)), u(t)) + (p(t, y(t)) - f(t, w(t)), v - g(t, u(t))), \ \forall v \in H, \ \forall t \in \Omega.$$

Thus u, w and y are a solution of (2.1). □

To obtain an approximate solution of (2.1) we can apply a successive approximation method to the problem of solving

$$u(t) \in F(t, u(t)), \ \forall t \in \Omega, \tag{3.2}$$

where

$$F(t, u(t)) = u(t) - g(t, u(t)) + J_{\alpha(t)}^{\varphi(\cdot, u(t))}(g(t, u(t)) - \alpha(t)(f(t, \widetilde{T}(t, u(t))) - p(t, \widetilde{A}(t, u(t))))).$$

Based on (3.1) and (3.2), we proceed our algorithm.

Suppose that $T, A : \Omega \times H \to \mathcal{F}(H)$ be two random fuzzy mappings satisfying the condition (I). Let $\widetilde{T}, \widetilde{A} : \Omega \times H \to CB(H)$ be two H-continuous random set-valued mappings induced by T, A, respectively, and $f, p, g : \Omega \times H \to H$ be continuous random operators. For any given measurable mapping $u_0 : \Omega \to H$, the set-valued mappings $\widetilde{T}(\cdot, u_0(\cdot)), \widetilde{A}(\cdot, u_0(\cdot)) : \Omega \to CB(H)$ are measurable by the lemma 3.1. Hence there exist measurable selection $w_0 : \Omega \to H$ of $\widetilde{T}(\cdot, u_0(\cdot))$ and $y_0 : \Omega \to H$ of $\widetilde{A}(\cdot, u_0(\cdot))$ by Himmelberg [17]. Let

$$u_1(t) = u_0(t) - g(t, u_0(t)) + J_{\alpha(t)}^{\varphi(\cdot, u_0(t))}(g(t, u_0(t)) - \alpha(t)(f(t, w_0(t)) - p(t, y_0(t)))).$$

It is easy to see that $u_1 : \Omega \to H$ is measurable. By Lemma 3.2, there exist measurable selections $w_1 : \Omega \to H$ of $\widetilde{T}(t, u_1(t))$ and $y_1 : \Omega \to H$ of $\widetilde{A}(t, u_1(t))$ such that

$$\|w_1(t) - w_0(t)\| \leq (1 + 1)\, H(\widetilde{T}(t, u_1(t)), \widetilde{T}(t, u_0(t))), \ \forall t \in \Omega,$$

$$\|y_1(t) - y_0(t)\| \leq (1 + 1)\, H(\widetilde{A}(t, u_1(t)), \widetilde{A}(t, u_0(t))), \ \forall t \in \Omega.$$

Letting

$$u_2(t) = u_1(t) - g(t, u_1(t)) + J_{\alpha(t)}^{\varphi(\cdot, u_1(t))}(g(t, u_1(t)) - \alpha(t)(f(t, w_1(t)) - p(t, y_1(t)))),$$

then $u_2 : \Omega \to H$ is measurable. By induction we can obtain our algorithm as following.

Algorithm 3.1. *Suppose that $T, A : \Omega \times H \to \mathcal{F}(H)$ be two random fuzzy mappings satisfying the condition (I). Let $\widetilde{T}, \widetilde{A} : \Omega \times H \to CB(H)$ be two H-continuous random set-valued mappings induced by T, A, respectively, and $f, p, g : \Omega \times H \to H$ be continuous random operators. For any given measurable mapping $u_0 : \Omega \to H$, we can get an algorithm for (2.1) as follows:*

$$\left.\begin{aligned} u_{n+1}(t) = u_n(t) - g(t, u_n(t)) + J_{\alpha(t)}^{\varphi(\cdot, u_n(t))}&(g(t, u_n(t)) \\ - \alpha(t)(f(t, w_n(t)) &- p(t, y_n(t)))), \\ w_n(t) \in \widetilde{T}(t, u_n(t)), \ y_n(t) &\in \widetilde{A}(t, u_n(t)), \\ \|w_{n+1}(t) - w_n(t)\| \leq (1 + (1+n)^{-1})\, H(\widetilde{T}&(t, u_{n+1}(t)), \widetilde{T}(t, u_n(t))), \\ \|y_{n+1}(t) - y_n(t)\| \leq (1 + (1+n)^{-1})\, H(\widetilde{A}&(t, u_{n+1}(t)), \widetilde{A}(t, u_n(t))), \end{aligned}\right\} \tag{3.3}$$

for any $t \in \Omega$ and $n = 0, 1, 2, \cdots$.

Remark 3.1. *The algorithm 3.1 includes several known algorithms of [6, 8-11, 16, 18, 20-22, 24, 25, 27, 31, 32, 34, 36, 37] as special cases.*

4. Existence and Convergence

Definition 4.1. A random operator $g : \Omega \times H \to H$ is said to be

(i) strongly monotone if there exists some a measurable function $\delta : \Omega \to (0, \infty)$ such that

$$(g(t,u_1) - g(t,u_2), u_1 - u_2) \geq \delta(t)\|u_1 - u_2\|^2, \ \forall u_i \in H, \ i = 1, 2, \ \forall t \in \Omega.$$

(ii) Lipschitz continuous if there exists some measurable function $\sigma : \Omega \to (0, \infty)$ such that

$$\|g(t,u_1) - g(t,u_2)\| \leq \sigma(t)\|u_1 - u_2\|, \ \forall u_i \in H, \ i = 1, 2, \ \forall t \in \Omega.$$

Definition 4.2. A random set-valued mapping $T : \Omega \times H \to CB(H)$ is said to be

(i) strongly monotone with respect to a random operator $f : \Omega \times H \to H$ if there exists some measurable function $\beta : \Omega \to (0, \infty)$ such that

$$(f(t,w_1) - f(t,w_2), u_1 - u_2) \geq \beta(t)\|u_1 - u_2\|^2, \forall t \in \Omega, \ \forall u_i \in H, \ \forall w_i \in T(t,u_i), \ i = 1, 2.$$

(ii) H-Lipschitz continuous if there exists some measurable function $\gamma : \Omega \to (0, \infty)$ such that

$$H(T(t,u_1), T(t,u_2)) \leq \gamma(t)\|u_1 - u_2\|, \ \forall u_i \in H, \ i = 1, 2.$$

Theorem 4.1. *Let $g : \Omega \times H \to H$ be strongly monotone and Lipschitz continuous random operator, $f, p : \Omega \times H \to H$ be Lipschitz continuous random operators, $T, A : \Omega \times H \to \mathcal{F}(H)$ be two random fuzzy mappings satisfying the condition (I). Let $\widetilde{T}, \widetilde{A} : \Omega \times H \to CB(H)$ be two random set-valued mappings induced by T, A respectively. Suppose that $\widetilde{T}$, $\widetilde{A}$ are H-Lipschitz continuous and $\widetilde{T}$ be strongly monotone with respect to f. Suppose there exists a measurable function $\xi : \Omega \to (0, +\infty)$ such that for each $x, y, z \in H$, $t \in \Omega$,*

$$\|J_{\alpha(t)}^{\partial\varphi(\cdot,x)}(z) - J_{\alpha(t)}^{\partial\varphi(\cdot,y)}(z)\| \leq \xi(t)\|x - y\|.$$

If the following conditions hold:

$$\left|\alpha(t) - \frac{\beta(t) + \epsilon(t)\mu(t)(k(t)-1)}{\eta^2(t)\gamma^2(t) - \epsilon^2(t)\mu^2(t)}\right| < \frac{\sqrt{(\beta(t) + (k(t)-1)\epsilon(t)\mu(t))^2 - l(t)}}{\eta^2(t)\gamma^2(t) - \epsilon^2(t)\mu^2(t)}, \tag{4.1}$$

$$\beta(t) > (1 - k(t))\epsilon(t)\mu(t) + \sqrt{l(t)}, \ \eta(t)\gamma(t) > \epsilon(t)\mu(t), \tag{4.2}$$

$$l(t) = (\eta^2(t)\gamma^2(t) - \epsilon^2(t)\mu^2(t))k(t)(2 - k(t)), \alpha(t)\mu(t)\epsilon(t) < 1 - k(t), \tag{4.3}$$

$$k(t) = \xi(t) + 2\sqrt{1 - 2\delta(t) + \sigma^2(t)}, \ k(t) < 1, \tag{4.4}$$

for all $t \in \Omega$, where $\beta(t)$ and $\delta(t)$ are strongly monotone coefficients of $\widetilde{T}$ and g, respectively, $\gamma(t)$ and $\mu(t)$ are H-Lipschitz coefficients of $\widetilde{T}$ and $\widetilde{A}$, respectively, $\sigma(t)$, $\eta(t)$ and $\epsilon(t)$ are the Lipschitz coefficients of g, f and p, respectively, then there exist measurable mappings u, w, $y : \Omega \to H$ such that (2.1) holds. Moreover,

$$u_n(t) \to u(t), \ w_n(t) \to w(t), \ y_n(t) \to y(t), \ n \to \infty,$$

where $\{u_n(t)\}$, $\{w_n(t)\}$ *and* $\{y_n(t)\}$ *are defined as in Algorithm 3.1.*

Proof. From (3.3), for any $t \in \Omega$, we have

$$\begin{aligned} \|u_{n+1}(t) - u_n(t)\| &= \|u_n(t) - u_{n-1}(t) - (g(t, u_n(t)) - g(t, u_{n-1}(t))) \\ &\quad + J_{\alpha(t)}^{\varphi(\cdot, u_n(t))}(h(t, u_n(t))) - J_{\alpha(t)}^{\varphi(\cdot, u_{n-1}(t))}(h(t, u_{n-1}(t)))\|, \end{aligned}$$

where

$$h(t, u_n(t)) = g(t, u_n(t)) - \alpha(t)(f(t, w_n(t)) - p(t, y_n(t))).$$

Also, we have

$$\begin{aligned} & \|J_{\alpha(t)}^{\varphi(\cdot, u_n(t))}(h(t, u_n(t))) - J_{\alpha(t)}^{\varphi(\cdot, u_{n-1}(t))}(h(t, u_{n-1}(t)))\| \\ \le\ & \|J_{\alpha(t)}^{\varphi(\cdot, u_n(t))}(h(t, u_{n-1}(t))) - J_{\alpha(t)}^{\varphi(\cdot, u_{n-1}(t))}(h(t, u_{n-1}(t)))\| \\ & + \|J_{\alpha(t)}^{\varphi(\cdot, u_n(t))}(h(t, u_n(t))) - J_{\alpha(t)}^{\varphi(\cdot, u_n(t))}(h(t, u_{n-1}(t)))\| \\ \le\ & \xi(t)\|u_n(t) - u_{n-1}(t)\| + \|h(t, u_n(t)) - h(t, u_{n-1}(t))\| \\ \le\ & \xi(t)\|u_n(t) - u_{n-1}(t)\| + \|u_n(t) - u_{n-1}(t) - \alpha(t)(f(t, w_n(t)) - f(t, w_{n-1}(t)))\| \\ & + \|u_n(t) - u_{n-1}(t) - (g(u_n(t)) - g(t, u_{n-1}(t)))\| \\ & + \alpha(t)\|p(t, y_n(t)) - p(t, y_{n-1}(t))\|, \ \ \forall t \in \Omega. \end{aligned}$$

That is

$$\begin{aligned} & \|u_{n+1}(t) - u_n(t)\| \\ \le\ & \xi(t)\|u_n(t) - u_{n-1}(t)\| + 2\|u_n(t) - u_{n-1}(t) - (g(t, u_n(t)) - g(t, u_{n-1}(t)))\| \\ & + \|u_n(t) - u_{n-1}(t) - \alpha(t)(f(t, w_n(t)) - f(t, w_{n-1}(t)))\| \\ & + \alpha(t)\|p(t, y_n(t)) - p(t, y_{n-1}(t))\|, \ \ \forall t \in \Omega. \end{aligned} \tag{4.5}$$

By Lipschitz continuity and strongly monotonicity of g, we obtain

$$\begin{aligned} & \|u_n(t) - u_{n-1}(t) - (g(t, u_n(t)) - g(t, u_{n-1}(t)))\|^2 \\ \le\ & (1 - 2\delta(t) + \sigma^2(t))\|u_n(t) - u_{n-1}(t)\|^2, \ \ \forall t \in \Omega. \end{aligned} \tag{4.6}$$

Also from H-Lipschitz continuity and strongly monotonicity of $\widetilde{T}$, and Lipschitz continuity of f, we have

$$\begin{aligned} & \|u_n(t) - u_{n-1}(t) - \alpha(t)(f(t, w_n(t)) - f(t, w_{n-1}(t)))\|^2 \\ \le\ & (1 - 2\beta(t)\alpha(t) + \alpha^2(t)\eta^2(t)(1 + n^{-1})^2\gamma^2(t))\|u_n(t) - u_{n-1}(t)\|^2, \ \ \forall t \in \Omega. \end{aligned} \tag{4.7}$$

By H-Lipschitz continuity of $\widetilde{A}$, Lipschitz continuity of p and (3.3), we know

$$\alpha(t)\|p(t, y_n(t)) - p(t, y_{n-1}(t))\| \le \alpha(t)\epsilon(t)(1 + n^{-1})\mu(t)\|u_n(t) - u_{n-1}(t)\|, \ \ \forall t \in \Omega. \tag{4.8}$$

Combining (4.5)-(4.8), we have

$$\|u_{n+1}(t) - u_n(t)\| \le \theta_n(t)\|u_n(t) - u_{n-1}(t)\|, \ \ \forall t \in \Omega,$$

where

$$\theta_n(t) := \xi(t) + 2\sqrt{1-2\delta(t)+\sigma^2(t)} + \sqrt{1-2\beta(t)\alpha(t)+\alpha^2(t)\eta^2(t)(1+n^{-1})^2\gamma^2(t)} + \alpha(t)\epsilon(t)(1+n^{-1})\mu(t).$$

Letting

$$\theta(t) := \xi(t)+2\sqrt{1-2\delta(t)+\sigma^2(t)}+\sqrt{1-2\beta(t)\alpha(t)+\alpha^2(t)\eta^2(t)\gamma^2(t)}+\alpha(t)\epsilon(t)\mu(t), \ \forall t \in \Omega.$$

we know that $\theta_n(t) \searrow \theta(t)$, for all $t \in \Omega$. It follows from (4.1)-(4.4) that $\theta(t) < 1$, for all $t \in \Omega$. Hence, for any $t \in \Omega$, $\theta_n(t) < 1$, for n sufficiently large. Therefore $\{u_n(t)\}$ is a Cauchy sequence and we can suppose that $u_n(t) \to u(t)$, for all $t \in \Omega$.

From (3.3), we get

$$\begin{aligned} \|w_n(t) - w_{n-1}(t)\| &\le (1+n^{-1})\gamma(t)\,\|u_n(t) - u_{n-1}(t)\|, \ \forall t \in \Omega, \\ \|y_n(t) - y_{n-1}(t)\| &\le (1+n^{-1})\mu(t)\,\|u_n(t) - u_{n-1}(t)\|, \ \forall t \in \Omega. \end{aligned}$$

So $\{w_n(t)\}$ and $\{y_n(t)\}$ are both Cauchy sequences. Let $w_n(t) \to w(t)$, $y_n \to y(t)$. Since $\{u_n(t)\}$, $\{w_n(t)\}$ and $\{y_n(t)\}$ are sequences of measurable mappings, we know that u, w, y : $\Omega \to H$ are measurable. Further, for any $t \in \Omega$, we have

$$\begin{aligned} d(w(t), \widetilde{T}(t, u(t))) &= \inf\{\|w(t) - z\| : z \in \widetilde{T}(t, u(t))\} \\ &\le \|w(t) - w_n(t)\| + d(w_n(t), \widetilde{T}(t, u(t))) \\ &\le \|w(t) - w_n(t)\| + H(\widetilde{T}(t, u_n(t)), \widetilde{T}(t, u(t))) \\ &\le \|w(t) - w_n(t)\| + \gamma(t)\|u_n(t) - u(t)\| \to 0. \end{aligned}$$

Hence, $w(t) \in \widetilde{T}(t, u(t))$, for all $t \in \Omega$. Similarly, $y(t) \in \widetilde{A}(t, u(t))$, for all $t \in \Omega$. □

Remark 4.1. *For a suitable choice of the mappings g, T, A, f, p and the function φ, we can obtain several known results [6, 8-11, 16, 18, 20-22, 24, 25, 27, 31, 32, 34, 36, 37] as special cases of the main result of this paper.*

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