Uniformly Integrable Family of Fuzzy Set-Valued Random Variables

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Abstract : In this paper, we discuss the uniformly integrable family of fuzzy set- valued random variables and its properties .We obtained the results that the sum of two uniformly integrable families of fuzzy random variables and the multiplication of an uniformly integrable family of fuzzy set- valued random variables by a real number are both uniformly integrable.

Keywords: Fuzzy set, fuzzy set-valued random variables, uniformly integrable family, fuzzy set-valued martingale.

1. Introduction

It is well known that set-valued functions and set-valued random variables have been used repeatedly in economics ([4], [8], [9]). The conditional expectation of set-valued random variable and set-valued martingale have been studied by F. Hiai, N. S. Papageorgiou, W. X. Zhang, e. t.. The discussions of fuzzy set-valued martingale have been originat in [1],[2]. This paper's purpose is to discuss uniformly integrable family of fuzzy set-valued random variables and its some properties, and to get some more deep results.

In section 2, some elemental notions and properties of fuzzy set-valued random variable are given.

In section 3, we will give some properties of uniformly integrable family of set-valued random variables, which will be used in section 4.

In section 4, there are some results of uniformly integrable family of fuzzy set-valued random variables.

2.Some notions of fuzzy set-valued random variable

Let X be a n-dimension Euclidean space and $(\Omega, \Im, P)$ be a complete probability measure

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space, $\{\Im_t\}_{t\in R_+}$ be a family of monotone increasing sub-σ-fields of $\Im$, $\Im_{\infty-} = \bigvee_{t\in R_+} \Im_t$, $\Im_{0-}$, $\Im_\infty$ be sub-σ-fields of $\Im$ and $\Im_{0-} \subset \Im_0$, $\Im_{\infty-} \subset \Im_\infty$ in this paper.

Let $\widetilde{F}_0(X)$ be the family of all fuzzy sets $\widetilde{A}: X \to [0,1]$ with properties:

(1) $\widetilde{A}$ is upper semicontinuous,

(2) $\widetilde{A}$ is fuzzy convex,

(3) $\widetilde{A}_\alpha$ is compact for every $\alpha \in (0,1]$,

where $\widetilde{A}_\alpha = \{x \in X: \widetilde{A}(x) \geq \alpha\}$ is the α-level set of $\widetilde{A}$.

A linear structure is defined in $\widetilde{F}_0(X)$ by

$(\widetilde{A}+\widetilde{B})(x) = \sup\{\alpha \in (0,1]: x \in \widetilde{A}_\alpha + \widetilde{B}_\alpha\}$,

$$(\lambda \widetilde{A})(x) = \begin{cases} \widetilde{A}(\lambda^{-1}x), & \text{if } \lambda \neq 0, \\ 0, & \text{if } \lambda = 0, \quad x \neq 0, \\ \sup\limits_{y\in X} \widetilde{A}(y), & \text{if } \lambda = 0, \quad x = 0, \end{cases}$$

for $\widetilde{A}, \widetilde{B} \in \widetilde{F}_0(X)$, $\lambda \in R$. It is easy to prove that $(\widetilde{A}+\widetilde{B})_\alpha = \widetilde{A}_\alpha + \widetilde{B}_\alpha$, $(\lambda \widetilde{A})_\alpha = \lambda \widetilde{A}_\alpha$ for every $\alpha \in [0,1]$.

Definition2.1. Let $\widetilde{F}: (\Omega, \Im) \to \widetilde{F}_0(X)$ be a mapping from $(\Omega, \Im)$ to $\widetilde{F}_0(X)$.

(1) $\widetilde{F}$ is called a fuzzy set-valued random variable, if

$\{\omega : \sup\limits_{y\in C} (\widetilde{F})(\omega)(x) \in B\} \in \Im$

for any closed subset C of X and Borel's subset B of [0,1], i.e. $B \in \mathcal{B}([0,1])$.

(2) $\widetilde{F}$ is called $\Im$ level measurable, if $\widetilde{F}_\alpha$ defined by $(\widetilde{F})_\alpha(\omega) = (\widetilde{F}(\omega))_\alpha$ for every $\omega \in \Omega$ is a set-valued random variable for every $\alpha \in (0,1]$.

The following two propositions are equivalent:

(1) $\widetilde{F}$ is a fuzzy set-valued random variable.

(2) $\widetilde{F}$ is $\Im$-level measurable.

Definition 2.2. Let $\widetilde{F}: (\Omega, \Im) \to \widetilde{F}_0(X)$ be a fuzzy set-valued random variable, $\widetilde{F}$ is called to

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be integrable bounded if there exists a nonnegative integrable function h such that

$$\left\|\widetilde{F}_\alpha(\omega)\right\| < h(\omega) \text{ for each } \alpha \in (0,1] \text{ and } \omega \in \Omega$$

Define

$$(\int_\Omega \widetilde{F}dp)(x) = \bigvee_{\alpha \in [0,1]} (\alpha \wedge I_{\int_\Omega \widetilde{F}_\alpha dP}(x))$$

where $\int_\Omega \widetilde{F}_0 dP = X$ and call $\int_\Omega \widetilde{F}dP$ to be integral of $\widetilde{F}$ on Ω.

Theorem 2.1. Let $\widetilde{F} : (\Omega, \Im) \to \widetilde{F}_0(X)$ be an integrable bounded fuzzy set-valued random variable, then $(\int_\Omega \widetilde{F}dP)_\alpha = \int_\Omega \widetilde{F}_\alpha dP$ for each $\alpha \in (0,1]$ and $\int_\Omega \widetilde{F}dP \neq \Phi$.

Proof: See[1] pp. 257-258.

Theorem 2.2. Let $\widetilde{F} : (\Omega, \Im) \to \widetilde{F}_0(X)$ be an integrable bounded fuzzy set-valued random variable, then for each sub-σ-field $\Im_1$ of $\Im$, there exists an unique $\Im_1$-measurable fuzzy set-valued variable $\widetilde{G}$ such that $\int_A \widetilde{F}dP = \int_A \widetilde{G}dP$ for each $A \in \Im_1$.

Proof: See [1] pp. 258-259.

Definition 2.3. Let $\widetilde{F} : (\Omega, \Im) \to \widetilde{F}_0(X)$ be an integrable bounded fuzzy set-valued random variable, $\Im_1$ be a sub-σ-field of $\Im$. $\widetilde{G} : (\Omega, \Im) \to \widetilde{F}_0(X)$ is called a conditional expectation of $\widetilde{F}$ with respect to the sub-σ-field $\Im_1$ of $\Im$, and is denoted as $E[\widetilde{F} | \Im_1]$, if $\widetilde{G}$ is a $\Im_1$-measurable integrable bounded fuzzy set-valued random variable satisfying the following condition:

$$\int_A \widetilde{F}dP = \int_A \widetilde{G}dP \quad \text{for each } A \in \Im_1.$$

The conditional expectation of $\widetilde{F}$ with respect to the sub-σ-field $\Im_1$ of $\Im$ is existent and a.s. unique and $(E[\widetilde{F}|\Im_1](\omega))_\alpha = E[\widetilde{F}_\alpha|\Im_1](\omega)$ a.s. by theorem 2.1, 2.2.

3. Some properties of unifomly integrable family of set-valued random variables

Definition 3.1. A family of set-valued variables $\{F_t : t \in T\}$ is said to be uniformly integrable if $\lim_{c\to+\infty}\int_{\{\omega:\|F_t(\omega)\|\geq c\}}\|F_t\|dP=0$ uniformly holds for $t\in T$, where $\|F_t(\omega)\|$=h({0}, $F_t(\omega)$} is the Hausdorff metric between {0} and $F_t(\omega)$.

Theorem 3.1. let $\mathfrak{R}$ be an uniformly integrable family of set-valued random variables, $\aleph$ be a family of set-valued random variables.If for each $F\in\aleph$,there exists a $G\in\mathfrak{R}$ such that $F\subseteq G$ a.s. ,then $\aleph$ is uniformly integrable.

The proof is simple.

Theorem 3.2. Let $\{F_t : t \in T\}$ and $\{G_t : t \in T\}$ are two unifirmly integrable families of set-valued random variables. Then $\{F_t \dot{+} G_t : t\in T\}$and $\{\lambda G_t : t\in T\}$ are also two uniformly integrable families of set-valued random variables, where

$\{F_t \dot{+} G_t\}(\omega)=\mathrm{cl}\{F_t(\omega)+G_t(\omega)\}, \lambda \in R$.

Proof: Since $\|F_t \dot{+} G_t\|\leq\|F_t\|+\|G_t\|$, $\|\lambda G_t\|\leq|\lambda|\cdot\|G_t\|$. But $\{\|F_t\|+\|G_t\| : t\in T\}$ and $\{\lambda\|G_t\| : t\in T\}$ are two uniformly integrable families of random variables, therefore $\{F_t \dot{+} G_t : t\in T\}$and $\{\lambda G_t : t\in T\}$are two uniformly integrable families of set-valued random variables by theorem 1.11 and corollary 1.14 in [5].

Theorem 3.3. Let F be an integrable bounded set-valued random variable, $\{\Im_t : t\in T\}$ be a family of sub-σ-fields of $\Im$. Then $\{E[F|\Im_t] : t\in T\}$ is uniformly integrable.

Proof: Since

$$0\leq\int_{\{\omega:\|E[F|\Im_t]\|\geq c\}}\|E[F|\Im_t]\|dP\leq\int_{\{\omega:E[\|F\||\Im_t]\geq c\}}E[\|F\||\Im_t]dP\to 0$$ holds uniformly for $t\in T$,

while $t\to\infty$. Then

$$\int_{\{\omega:\|E[F|\Im_t]\|\geq c\}}\|E[F|\Im_t]\|dP\to 0$$ holds uniformly for $t\in T$, while $t\to\infty$.

Thus $\{E[F|\Im_t] : t\in T\}$ is uniformly integrable.

Theorem 3.4. Let $\{F_t : t\in T\}$ be an uniformly integrable family of set-valued random variables and x be any point in X. Then $\{\sigma(x, F_t) : t\in T\}$ is an uniformly integrable family of random variables,where

$$\sigma(x, F_t(\omega)) = \sup_{y \in F_t(\omega)} (x, y),$$

where (x , y) is the inner product of x and y.

Proof : Since

$$\sigma(x, F_t(\omega)) = \sup_{y \in F_t(\omega)} (x, y) \le \sup_{y \in F_t(\omega)} \|x\| \cdot \|y\| = \|x\| \cdot \|F_t(\omega)\|.$$

Then $\{ \sigma(x, F_t) : t \in T \}$ is an uniformly integrable family of random variables by theorem 1.11 in[5].

Theorem 3.5. Let $\{F_t : t \in T\}$ and $\{G_t : t \in T\}$ be two uniformly integrable families of set-valued random variables. Then $\{h(F_t, G_t) : t \in T\}$ is uniformly integrable.

Proof : Since

$$h(F_t(\omega), G_t(\omega)) \le h(\{0\}, F_t(\omega)) + h(\{0\}, G_t(\omega)) = \|F_t(\omega)\| + \|G_t(\omega)\|,$$

Then $\{h(F_t, G_t) : t \in T\}$ is uniformly integrable by theorem 1.11 and corollary 1.14 in[5].

4. The uniformly integrable family of fuzzy set-valued random variables

Definition 4.1. A family of fuzzy set-valued random variables $\{ \widetilde{F}_t : t \in T \}$ is said to be uniformly integrable if $\{ (\widetilde{F}_t)_\alpha : t \in T, \ \alpha \in (0,1] \}$ is an uniformly integrable family of set-valued random variables.

Theorem 4.1. Let $\Re$ and $\aleph$ be two families of fuzzy set-valued random variables and $\aleph$ be uniformly integrable. Then $\Re$ is uniformly integrable if there exists $\widetilde{F} \in \aleph$ for each $\widetilde{G} \in \Re$ such that $\widetilde{G} \subseteq \widetilde{F}$ a.s.

Proof: By assumption, $\{ (\widetilde{F})_\alpha : \widetilde{F} \in \aleph, \ \alpha \in (0,1] \}$ is an uniformly integrabe family of set-valued variables,and there exists $\widetilde{F} \in \aleph$ for each $\widetilde{G} \in \Re$ such that $\widetilde{G} \subset \widetilde{F}$ a.s..Then there exists $(\widetilde{F})_\alpha \in \{ (\widetilde{F})_\alpha, \ \widetilde{F} \in \aleph, \ \alpha \in (0,1] \}$ for each $(\widetilde{G})_\alpha \in \{ (\widetilde{G})_\alpha, \ \widetilde{G} \in \Re, \ \alpha \in (0,1] \}$ such that $(\widetilde{G})_\alpha \subseteq (\widetilde{F})_\alpha$ a.s.. Then $\{ (\widetilde{G})_\alpha, \ \widetilde{G} \in \Re, \ \alpha \in (0,1] \}$ is uniformly integrable by theorem3.1. Therefore $\Re$ is uniformly integrable.

Theorem 4.2. Let $\{\widetilde{F}_t : t \in T\}$ and $\{\widetilde{G}_t : t \in T\}$ be two uniformly integrable families of fuzzy set-valued random variables. Then $\{\widetilde{F}_t + \widetilde{G}_t : t \in T\}$ and $\{\lambda \widetilde{F}_t : t \in T\}$ are two uniformly integrable families of fuzzy set-valued random variables.

Proof: Since $\{(\widetilde{F}_t)_\alpha : t \in T, \alpha \in (0,1]\}$ and $\{(\widetilde{G}_t)_\alpha : t \in T, \alpha \in (0,1]\}$ are two uniformly integrable families of set-valued random variables by assumption and definition 4.1. Then $\{(\widetilde{F}_t)_\alpha \dot{+} (\widetilde{G}_t)_\alpha : t \in T, \alpha \in (0,1]\}$ is an uniformly integrable families of set-valued random variables by theorem 3.2. Because

$$(\widetilde{F}_t + \widetilde{G}_t)_\alpha = (\widetilde{F}_t)_\alpha + (\widetilde{G}_t)_\alpha \subseteq (\widetilde{F}_t)_\alpha \dot{+} (G_t)_\alpha \text{ for any } t \in T,\ \alpha \in (0,1],$$

therefore $\{(\widetilde{F}_t + \widetilde{G}_t)_\alpha : t \in T,\ \alpha \in (0,1]\}$ is an uniformly integrable families of set-valued random variables by theorem 3.1 . Further $\{\widetilde{F}_t + \widetilde{G}_t : t \in T\}$ is an uniformly integrable family of fuzzy set-valued random variables by definition 4.1.

Similarly it can be proved that $\{\lambda \widetilde{F}_t : t \in T\}$ is an uniformly integrable family of fuzzy set-valued random variables.

Theorem 4.3. Let $\widetilde{F}$ be an integrably bounded fuzzy set-valued random variable. $\{\Im_t\}_{t\in T}$ be a family of sub-σ-fields of $\Im$. Then $\{E[\widetilde{F}|\Im_t] : t \in T\}$ is an uniformly integrable family of fuzzy set-valued random variables.

Proof: Since $(E[\widetilde{F}|\Im_t])_\alpha = E[\widetilde{F}_\alpha|\Im_t]$ a.s. for each $\alpha \in [0,1]$, $t \in T$ by definition 2.3. Furthermore,

$$\int_{[\|(E[\widetilde{F}|\Im_t])_\alpha\| \ge c]} \|(E[\widetilde{F}|\Im_t])_\alpha\| dP$$

$$= \int_{[\|E[\widetilde{F}_\alpha|\Im_t]\| \ge c]} \|E[\widetilde{F}_\alpha|\Im_t]\| dP$$

$$\le \int_{[E[\|\widetilde{F}_\alpha\| |\Im_t] \ge c]} E[\|\widetilde{F}_\alpha\| |\Im_t] dP$$

$$\le \int_{[\|\widetilde{F}_\alpha\| < \delta]} \|\widetilde{F}_\alpha\| I_{[E[\|\widetilde{F}_\alpha\| |\Im_t] \ge c]} dP + \int_{[\|\widetilde{F}_\alpha\| \ge \delta]} \|\widetilde{F}_\alpha\| dP$$

$$\le \delta\ P\{E[\|\widetilde{F}_\alpha\| |\Im_t] \ge c\} + \int_{[\|\widetilde{F}_\alpha\| \ge \delta]} \|\widetilde{F}_\alpha\| dP$$

$$\leq \frac{\delta}{c} E[\|\widetilde{F}_\alpha\|] + \int_{[\|\widetilde{F}_\alpha\| \geq \delta]} \|\widetilde{F}_\alpha\| dP .$$

Since $\widetilde{F}$ is integrably bounded, then $\sup_{\alpha \in (0,1]} E[\|\widetilde{F}_\alpha\|] < \infty$ and $\{\|\widetilde{F}_\alpha\| : \alpha \in (0,1]\}$ is uniformly integrable .Thus for any positive number ε, we can find a positive number δ such that $\int_{[\|\widetilde{F}_\alpha\| \geq \delta]} \|\widetilde{F}_\alpha\| dP < \frac{\varepsilon}{2}$ for each $\alpha \in (0,1]$. Thus

$$\int_{[\|(E[\widetilde{F}|\Im_t])_\alpha\| \geq c]} \|(E[\widetilde{F}|\Im_t])_\alpha\| dP < \varepsilon$$

for every $\alpha \in (0,1]$, $t \in T$, when $c \geq \frac{2\delta}{\varepsilon} \sup_{\alpha \in (0,1]} E[\|\widetilde{F}_\alpha\|]$. Therefore $\{E[[\widetilde{F}_\alpha|\Im_t] : \alpha \in (0,1], t \in T\}$ is uniformly integrable. Hence $\{E[\widetilde{F}|\Im_t] : t \in T\}$ is an uniformly integrable family of fuzzy set-valued random variables.

Definition 4.2. Let $\{\Im_t\}_{t \in R_+}$ be a family of sub-σ-fields of $\Im$, $\{\widetilde{F}_t\}_{t \in R_+}$ be a $\{\Im_t\}_{t \in R_+}$-adapted integrably bounded fuzzy set-valued stochastic process, i.e. $\widetilde{F}_t$ is $\Im_t$-measurable and integrably bounded for each $t \in R_+$. $\{\widetilde{F}_t ; \Im_t\}_{t \in R_+}$ is said to be a fuzzy set-valued martingale (resp. supermartingale, submartingale) if

$$E[\widetilde{F}_t|\Im_s] = \widetilde{F}_s \text{ (resp. } \subseteq \widetilde{F}_s \;\; \supseteq \widetilde{F}_s \text{) a.s. for } s<t,\ s, t \in R_+ .$$

If there exists a $\Im_\infty$-measurable integrably bounded fuzzy set-valued random variable $\widetilde{F}_\infty$ such that

$$E[\widetilde{F}_\infty|\Im_t] = \widetilde{F}_t \text{ (resp. } \subseteq \widetilde{F}_t ,\ \supseteq \widetilde{F}_t \text{) a.s.},$$

$\{\widetilde{F}_t ; \Im_t\}_{t \in R_+}$ is said to be a closed fuzzy set-valued martingale (resp. supermartingale, submartingale) at the right side, and $\widetilde{F}_\infty$ close $\{\widetilde{F}_t ; \Im_t\}_{t \in R_+}$ from the right side.

Corollary 4.1. Let $\{\widetilde{F}_t ; \Im_t\}_{t \in R_+}$ be a closed fuzzy set-valued martingale or a closed fuzzy set-valued submartingale at the right side, and $\widetilde{F}_\infty$ close $\{\widetilde{F}_t ; \Im_t\}_{t \in R_+}$ from the right side. Then $\{\widetilde{F}_t\}_{t \in R_+}$ is uniformly integrable.

Proof: This corollary is obtained immediately by theorem 4.3 and theorem 4.1.

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