Fuzzy N(2,0) algebra

Deng Fang'an
(Department of Mathematics,Hanzhong Teacher's College, ShaanXi, 723000)
Jiang Zhao Lin
(Department of Mathematics, Linyi Teacher's College, Shandong, 276005)
Xu Yang
(Department of application of Mathematics, SouthWest Jiao tong Univesity, ChenDu, Sichuan, 610031)

Abstract: The purpose of this paper is introduce and investigate the fuzzy N(2,0) algebra, the fuzzy N-ideal and the fuzzy partial group of N(2,0) algebra are studied.

Keywords: N(2,0) algebra ; Fuzzy subalgebra; fuzzy N- ideal; fuzzy partial group

1. Introduction

N(2,0) algebra is a Commutative Monoid,the relation between its proper subclass —— Strong N(2,0) algebra and lattice implication algebra was studied.[1] In this paper, we define the fuzzy N(2,0) algebra and the fuzzy N-ideal of fuzzy N(2,0) algebra, and study some of their basic properties. Moreover, we get out of the restrict of the theory for algebraic structures wearker than groups and which depend on the idempotents such as clifford, inverse, and completely regular semigroups, we define partial element and partial inverse of N(2,0) algebra.

2. Preliminaries

Definition 2.1 Let S be a set with a constant 0, and the binary operation $*$ subject to:

(N_1) $x*(y*z) = z*(x*y)$

(N_2) $(x*y)*z = y*(x*z)$

(N_3) $0*x = x$

for any $x,y,z \in S$, then $(S,*,0)$ is said to be a N(2,0) algebra.

Remark. Let $(S,*,0)$ be a N(2,0) algebra, then the following identities hold for any $x,y,z \in S$,

(1) $x*y = y*x$

(2) $(x*y)*z = x*(y*z)$

(3) $x*(y*z) = y*(x*z)$, $(x*y)*z = (x*z)*y$

(4) 0 is Unit element.

Definition 2.2 Let S be a set. A fuzzy set in S is a function $\mu : S \to [0,1]$.

Definition 2.3 Let μ be a fuzzy set in S. For $\alpha \in [0,1]$, the set $\mu_{\alpha}=\{x \in S,\ \mu(x) \geqslant \alpha\}$ is called a level subset of μ.

Defintion 2.4 Let S be a $N(2,0)$ algebra, a function $\mu: S \to [0,1]$ is said to a fuzzy subalgebra of $N(2,0)$ algebra, if for any $x,y \in S$,

$$\mu(x*y) \geqslant min\ \{\mu(x), \mu(y).$$

Theorem 2.5 Let S be a $N(2,0)$ algebra, $\mu: S \to [0,1]$ is a fuzzy subalgebra of S, if and only if $\lambda \in [0,1]$,

$$\mu_{\lambda}=\{x|x \in S,\ \mu(x) \geqslant \lambda\}$$ is subalgebra of S, if $\mu_{\lambda} \neq \phi$.

Proof. Straightforward.

Let S be a $N(2,0)$ algebra and $x \in S$. For any $t,r \in [0,1]$, $M(t,r)$ will denote $min(t,r)$. The fuzzy subset λ_x of S defined by

$$\lambda_x(g) = M(\lambda(g*x), \lambda(0)) \quad \forall g \in S.$$

is called the fuzzy coset determined by x and λ.

Theorem 2.6 Let λ be a fuzzy subset of S, S_{λ}, the set of all fuzzy cosets of λ in S is a $N(2,0)$algebra under the multiplication defined as follows:

$$\lambda_x(g) \odot \lambda_y(g) = \lambda_{x*y}(g) \qquad \forall x,\ y,\ g \in S$$

Proof. we show that the composition are well defined. For any $g \in S$

$$\lambda_o(g) \odot \lambda_x(g) = M(\lambda(g*0), \lambda(0)) \odot M(\lambda(g*x), \lambda(0))$$
$$= \lambda_{o*x}(g) = M(\lambda(g*(0*x)), \lambda(0)$$
$$= M(\lambda(g*x), \lambda(0)) = \lambda_x(g)$$

hence, $\lambda_o(g)$ is unit for $\odot$ operator.

and $\lambda_x \odot \lambda_y = M(\lambda(g*x), \lambda(0)) \odot M(\lambda(g*y), \lambda(0))$

$\lambda_{x*y} = M(\lambda(g*(x*y)), \lambda(0))$

so $M(\lambda(g*(x*y)), \lambda(0)) = M(\lambda(g*x), \lambda(0)) \odot M(\lambda(g*y), \lambda(0))$

Since $\lambda_{x*(y*z)}(g)$

$$= M(\lambda(g*x*y*z))), \lambda(0))$$
$$= M(\lambda(g*z*(x*y)), \lambda(0))$$
$$= M(\lambda(g*z), \lambda(0)) \odot M(\lambda(g*x*y)), \lambda(0))$$
$$= M(\lambda(g*z), \lambda(0)) \odot ((M(\lambda(g*x), \lambda(0)) \odot (M(\lambda(g*y), \lambda(0)))$$
$$= \lambda_z \odot (\lambda_x \odot \lambda_y)*(g)$$

on the other hand, $M(\lambda(g*(x*(y*z))), \lambda(0)) = \lambda_x \odot (\lambda_y \odot \lambda_z)$

Thus, $\lambda_x \odot (\lambda_y \odot \lambda_z) = \lambda_z \odot (\lambda_x \odot \lambda_y)$

Similarly $(\lambda_x \odot \lambda_y) \odot \lambda_z = \lambda_y \odot (\lambda_x \odot \lambda_z)$

So, $(S_{\lambda}, \odot, \lambda_o)$ is a $N(2,0)$ algebra.

3. Fuzzy N-ideal of N(2,0) algebra

Definition 3.1 Let S be a $N(2,0)$ algebra, a map $A: S \to [0,1]$ is called fuzzy N-ideal of S, if for all $x,y \in S$,

$$A(x*(x*y)) \geqslant A(x)$$

Lemma 3.2 If A be a fuzzy N-ideal of S, then $A(x) \geqslant A(0)$ is hold for any $x \in S$.

Proof. For any $x \in S$, Since

$$A(x)=A(0*(0*x)) \geqslant A(0)$$

hence $A(x) \geqslant A(0)$ for any $x \in S$.

Theorem 3.3 Let S be a $N(2,0)$algebra, $E(S)$is a idempotents set of S. A is a fuzzy subset of $E(S)$, then A is a fuzzy N-ideal of

$E(S)$, *if only if* A *is a fuzzy subalgebra of* $E(S)$.

Proof. Let $x,y \in E(S)$, then $x*x = x$, $y*y = y$,

that is $x*y = x*(x*y)$ implies

$$A(x*y) = A(x*(x*y)) \geqslant A(x), \text{ and } y*x = x*y = y*(y*x), \text{ that is}$$

$$A(x*y) = A(y*(y*x) \geqslant A(y)$$

so $A(x*y) \geqslant min(A(x),A(y))$.

Conversely, if A is a subalgebra of $E(S)$, then

$$A(x*y) \geqslant min(A(x),A(y)), \text{ that is}$$

$$A(x*y) = A(x*(x*y)) \geqslant min(A(x),A(y))$$

and $A(x*y) = A(y*x)=A(y*(y*x) \geqslant min(A(x),A(y))$

hence, $A(x*(x*y)) \geqslant A(x)$ or $A(y*(y*x)) \geqslant A(y)$

Therefore A is a fuzzy N-ideal of $E(S)$.

Example 3.4 Let $S=\{0,a,b\}$ be the $N(2,0)$ algebra with the following mutiplication table:

*	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

where $E(S)=\{0,a,b\}$, Let $\mu: S \to [0,1]$ be defined by $\mu(0)=1/4$, $\mu(a)=\mu(b)=1$, then μ is a fuzzy N-ideal of S. Also μ is a fuzzy subalgebra of S.

4. Fuzzy subpartial groups of N(2,0) algebra

Throughout S ia a $N(2,0)$ algebra, $E(S)$ is the set of all idempotents is S.

Definition 4.1 An element $ex \in S$ is called a partial identity for $x \in S$, if

$$e_x*x = x \qquad (e_x \neq 0)$$

Obvious, for any $x \in S$, $e_x \in E(S)$.

Definition 4.2 If x has a partial identity e_x, then $y \in S$ is called a partial inverse of x if

(i) $x*y = e_x$ (ii) $e_x*y = y$

The partial inverse of x, if exists, is unique and is dnoted by x^{-1}.

Theorem 4.3[2] S is a Completely regular semigroup if and only if every $x \in S$ has a partial inverse and a partial identity.

Theorem 4.4[2] A Completely regular semigroup is a partial group if and only if $E(S)$ is Commutive.

From theorem 4.3 and Theorem 4.4 we have:

Theorem 4.5 If every $x \in S$ has a partial inverse and a partial identity, then S is a partial group.

From now on S is a partial group unless stated otherwise.

Theorem 4.6 Let A be a fuzzy subset of S, if for every $x \in S$

(i) A is a N-ideal of S, then $A(x) \geqslant A(e_x)$

(ii) A is a subalgebra of S, then $A(e_x)\geqslant min(A(x),A(x^{-1}))$

Proof. For any $x\in S$, we have

(i) $A(x) = A(e_x*(e_x*x))\geqslant A(e_x)$

(ii) $A(e_x) = A(x*x^{-1})\geqslant min(A(x),A(x^{-1}))$

Definition 4.7[2] Let $(S,*,0)$ be a $N(2,0)$ algebra and $\varepsilon : S\to S$, $x\to x^{\varepsilon}$ be a unary operation. Then $(S,*,\varepsilon)$ is a partial monoid if the following are satisfied, $\forall x,y\in S$.

(PM_1) x^{ε} is an idempotent

(PM_2) $(x^{\varepsilon})^{\varepsilon} = x^{\varepsilon}$

(PM_3) $x^{\varepsilon}*x = x$

(PM_4) $(x*y)^{\varepsilon} = x^{\varepsilon}*y^{\varepsilon}$

Definition 4.8 A function $\mu: S\to[0,1]$ is called a fuzzy subpartial monoid of the partial moniod $(S,*,\varepsilon)$ if for all $x, y\in S$, we have

(i) $\mu(x*y)\geqslant min(\mu(x),\mu(y))$

(ii) $\mu(x^{\varepsilon}) = \mu(x)^{\varepsilon}$, $\mu(x)^{\varepsilon}$ is the partial identity of $\mu(x)$ in $[0,1]$.

Definition 4.9 A function $\mu: S\to[0,1]$ is called a fuzzy subpartial group of the partial group S if for $x, y\in S$, we have

(i) $\mu(x*y)\geqslant min(\mu(x),\mu(y))$

(ii) $\mu(x^{-1}) = \mu(x)$

(iii) $\mu(e_x) = \mu(x)^{\varepsilon}$

where e_x is the partial identity of x in S and $\mu(x)^{\varepsilon}$ is the partial identity of $\mu(x)$ in $[0,1]$.

Theorem 4.10 A fuzzy subset μ of S is a fuzzy subpartial group of S if and only if

$$\mu(e_x) = \mu(x)^{\varepsilon}, \quad \text{and} \quad \mu(x*y^{-1})\geqslant min(\mu(x),\mu(y)),$$

for all $x,y\in S$.

Proof. Clear, if μ is a fuzzy subpartial group of S, then

$$\mu(x*y^{-1})\geqslant min(\mu(x),\mu(y^{-1})) = min(\mu(x),\mu(y)),$$

and $\mu(e_x) = \mu(x)^{\varepsilon}$.

Conversely, if $\mu(e_x) = \mu(x)^{\varepsilon}$ and $\mu(x*y^{-1})\geqslant min(\mu(x),\mu(y))$, for all $x,y\in S$, then

$$\mu(x^{-1}) = \mu(e_x*x^{-1})\geqslant min(\mu(e_x),\mu(x))=min(\mu(x)^{\varepsilon},\mu(x))=\mu(x),$$

$$\mu(x) = \mu(e_x*(x^{-1})^{-1})\geqslant min(\mu(e_x),\mu(x^{-1})$$

$$= min(\mu(x^{-1})^{\varepsilon},\mu(x^{-1})) = \mu(x^{-1})$$

hence $\mu(x) = \mu(x^{-1})$. Also

$$\mu(x*y) = \mu(x*(y^{-1})^{-1})\geqslant min(\mu(x),\mu(y^{-1})) = min(\mu(x),\mu(y)).$$

Therefore μ is a fuzzy subpartial group of s.

Definition 4.11 For any $x, y\in S$, we defined

$$(x,y)\in R, \text{ iff } x*e_y = x \text{ and } y*e_x = y$$

Theorem 4.12 R is a congruence relation on S.

Proof. Clear R is reflexive, symmetric. For any $x, y, z\in S$,

From $x*e_y = x$ and $y*e_x = y$

$y*e_z = y$ and $z*e_y = z$

so $x*e_y = x*(y*y^{-1}) = x$ $\quad x*(y*e_z)*y^{-1} = x$

$$x*e_y*e_z = x \qquad x*e_z = x$$

$$z*e_x = z*(x*x^{-1}) = z*(x*e_z)*x^{-1}) = z*e_y*e_x = z*(y*y^{-1})*e_x$$
$$= z*(y*e_x)*y^{-1} = z*y*y^{-1} = z*e_y = z$$

hence $(x, z) \in R$. *Therefore, R is a epuivalent relation on S.*

Further, $\forall x_1, x_2, y_1, y_2 \in S$, *if* $(x_1,x_2) \in R$, *and* $(y_1,y_2) \in R$, *then*

$$((x_1*y_1, x_2*y_2) \in R.$$

where

$$\begin{aligned}(x_1*x_2)*e_{y_1*y_2} &= (x_1*e_{y_1})*(x_2*e_{y_2})*e_{y_1}*y_2 \\ &= (x_1*e_{y_1})*(x_2*e_{y_2})*(y_1*y_2)^{-1}*(y_1*y_2) \\ &= (x_1*e_{y_1})*(x_2*e_{y_2})*y_1^{-1}*y_2^{-1}*y_1*y_2 \\ &= (x_1*(e_{y_1}*e_{y_2}))*(x_2*(e_{y_2}*e_{y_2})) \\ &= (x_1*e_{y_1})*(x_2*e_{y_2}) \\ &= x_1*x_2\end{aligned}$$

similarly, $(y_1*y_2)*e_{x_1}*x_2 = y_1*y_2$, *so R is a congruence on S.*

For a congruence relation "R", we defined by the following:

$$[x] = \{\ y \mid y \in S \ \text{and}\ (y, x) \in R\ \}$$
$$S/_R = \{\ [x] \mid x \in S\}$$

In $S/_R$, *the binary defined by the following:*

$$\forall\ [x],[y] \in S/_R \qquad [x]☆[y] = [x*y]$$

Then one may easily verify that $(S/_R,☆)$ *is a N(2,0) algebra, also called it is a quotient N(2,0) algebra.*

Reference

[1] *Xu Yang, Deng Fang'an, N(2,0) algebra and Lattice implication Algebra (to apper).*

[2] *A.M.Abd-Allah, R.A.K.Omar, Fuzzy partial group, Fuzzy sets and systems, 82 (1996) 369–374.*

[3] *Ju Pil Kim, Deok Rak Bae, Fuzzy Congruence in groups, Fuzzy sets and systems, 85 (1997) 115–120.*

[4] *Xu Yang, Qin KeYun, Fuzzy Lattice implication Algebras. The Journal of Southevest Jiaotong University, Vol 30, №.2 (1995), 121–127.*