# ON CONTAINWISE REGULARITY

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**Abstract** In this paper, the multiplication of containwise regular spaces is discussed. Furthermore, the results that the containwise regularity in an induced space is equivalent to the regularity in its underlying space under the condition of closure preserving and the containwise regularity is a good extension in the sense of Lowen are obtained.

**Keywords** LF topological spaces, induced spaces, containwise regularity.

The regular separation is one of the most important conceptions in topology. Because there is gradated structure in LF topological spaces which be not in general topological spaces, there are varied LF regular separations[1] which have the research of regularity rich. One important and interesting subject of this research is to discuss the relations between two kinds of regularity, the one is in an induced space and the other in its underlying space. Some LF regularity in lattice-valued induced spaces were discussed in [1][2][3]. The principal job in this paper is to study the relations between the containwise regularity in a lattice valued induced space and the regularity in its underlying space. The results in this paper show us that the containwise regularity is a good extension in the sense of Lowen and under some condition the two kinds of regularity stated above are equivalent.

**Definition** 1[2] Let $(L^X, \delta)$ be an LF topological space. $(L^X, \delta)$ is called a containwise regular space if for each open set $U$ and each LF point $x_\lambda$ in $U$ there is an open set $V$ such that $x_\lambda \leqslant V \leqslant V^- \leqslant U$ .

**Definition 2** Let $(L^X, \delta)$ be an LF topological space. Called $(L^X, \delta)$ a closure preserving space for the family of open sets, if $(\bigvee_{t \in T} U_t)^- = \bigvee_{t \in T} U_t^-$ for every family of open sets in $(L^X, \delta)$ .

At the first of the following, we shall discuss the multiplication of containwise regular spaces.

**Theorem 1** Let $(L^X, \delta)$ be the product space of $\{(L^{X_t}, \delta_t)\}_{t \in T}$ .

(1) If $(L^X, \delta)$ is a containwise regular space, then so is $(L^{X_t}, \delta_t)$ for every $t \in T$ .

(2) If $(L^{X_t}, \delta_t)$ is a containwise regular space for each $t \in T$ and $(L^X, \delta)$ is a closure preserving space for the family of open sets, then so is $(L^X, \delta)$ .

**Proof** (1) From that the containwise regularity is a topological unchanged property and the project mapping is a continuous order homeomorphism, (1) holds.

(2) Let $U \in \delta, x_\lambda \leqslant U$ . Because there is a base of $\delta$ consisted of the open set $P_{t_1}^{-1}(U_{t_1}) \wedge \cdots \wedge P_{t_k}^{-1}(U_{t_k})$ where $U_{t_i} \in \delta_{t_i}, t_i \in T, k \in N$ , so for each $\delta \in \beta^*(\lambda)$, there exists $U_\delta = P_{t_1}^{-1}(U_{t_1}) \wedge \cdots \wedge P_{t_{k(\delta)}}^{-1}(U_{t_{k(\delta)}}) \leqslant U$ such that $x_\delta \leqslant U_\delta$ . Thus for each $i \leqslant k(\delta), x_\delta \leqslant P_{t_i}^{-1}(U_{t_i})$ . In this case, $(x_{t_i})_\delta = (x_\delta)_{t_i} = P_{t_i}(x_\delta) \leqslant U_{t_i}$. From that $(L^{X_i}, \delta_{t_i})(i \leqslant k(\delta))$ is a containwise regular space, we know that there exists $V_{t_i} \in \delta_{t_i}$ such that $(x_{t_i})_\delta \leqslant V_{t_i} \leqslant V_{t_i}^- \leqslant U_{t_i}$ . Since $x_\delta = P_{t_i}^{-1}((x_{t_i})_\delta) \leqslant P_{t_i}^{-1}(V_{t_i}) \leqslant P_{t_i}^{-1}(V_{t_i}^-) \leqslant P_{t_i}^{-1}(U_{t_i}), x_\delta \leqslant \wedge_{i=1}^{k(\delta)} P_{t_i}^{-1}(V_{t_i}) \leqslant (\wedge_{i=1}^{k(\delta)} P_{t_i}^{-1}(V_{t_i}))^- \leqslant \wedge_{i=1}^{k(\delta)} P_{t_i}^{-1}(V_{t_i}^-) \leqslant \wedge_{i=1}^{k(\delta)} P_{t_i}^{-1}(U_{t_i}) = U_\delta$ . Taking $V_\delta = \wedge_{i=1}^{k(\delta)} P_{t_i}^{-1}(V_{t_i})$ , then $x_\delta \leqslant V_\delta \leqslant V_\delta^- \leqslant U_\delta$ . From the condition listed in this theorem, $x_\lambda \leqslant \bigvee_{\delta \in \beta^*(\lambda)} V_\delta \leqslant$

$(\bigvee_{\delta\in\beta^*(\lambda)}V_\delta)^- = \bigvee_{\delta\in\beta^*(\lambda)}V_\delta^- \leqslant \bigvee_{\delta\in\beta^*(\lambda)}U_\delta \leqslant U$. Taking $V = \bigvee_{\delta\in\beta^*(\lambda)}V_\delta$, then $x_\lambda \leqslant V \leqslant V^- \leqslant U$. Therefore $(L^X,\delta)$ is a containwise regular space.

At the second of the following, we shall study the relations between the containwise regularity in an induced space and the regularity of its underlying space.

**Theorem 2** Let $(L^X, \omega_L(\mathscr{T}))$ be an induced space generated by $(X, \mathscr{T})$.

(1) If $(L^X, \omega_L(\mathscr{T}))$ is a containwise regular space, then $(X, \mathscr{T})$ is a regular space.

(2) If $(X, \mathscr{T})$ is a regular space and $(L^X,\delta)$ is a closure preserving space for the family of open sets, then $(L^X, \omega_L(\mathscr{T}))$ is a containwise regular space.

**Proof** (1) $\forall\, U \in \mathscr{T}$, $x \in U$, we have $\chi_U \in \omega_L(\mathscr{T})$. Taking arbitrary $\lambda \in L\backslash\{0\}$, then $x_\lambda \leqslant \chi_U$. Since $(L^X, \omega_L(\mathscr{T}))$ is a containwise regular space, there exists $V \in \omega_L(\mathscr{T})$ such that $x_\lambda \leqslant V \leqslant V^- \leqslant \chi_U$. Since $(L^X, \omega_L(\mathscr{T}))$ is induced, $V_{[\lambda]} = (V^\circ)_{[\lambda]} = \bigcap_{\delta\in\beta(\lambda)}(V_{[\delta]})^\circ$. Taking arbitrary $\delta \in \beta(\lambda)$, by $x_\lambda \leqslant V$, we have $x \in V_{[\lambda]} \subseteq (V_{[\delta]})^\circ \subseteq ((V^-)_{[\delta]})^\circ \subseteq (V^-)_{[\delta]} \subseteq U$. So $x \in V_{[\lambda]} \subseteq (V_{[\delta]})^\circ \subseteq (V_{[\delta]})^{\circ-} \subseteq ((V^-)_{[\delta]})^- = (V^-)_{[\delta]} \subseteq U$. Therefore $(X, \mathscr{T})$ is a regular space.

(2) For each $U \in \omega_L(\mathscr{T})$ and $x_\lambda \leqslant U$, by Theorem 2. 11. 20 in [2], there exists $L_1 \subseteq L$ such that $U = \bigvee_{r\in L_1} r\chi_{G_r}$ in which $G_r \in \mathscr{T}$ for each $r \in L_1$. So for every $\delta \in \beta^*(\lambda)$, there exists $r(\delta) \in L_1$ such that $x_\delta \leqslant r(\delta)\chi_{G_{r(\delta)}}$, from which we know that $x \in G_{r(\delta)}$ and $\delta \leqslant r(\delta)$. Since $(X, \mathscr{T})$ is a regular space, there exists $A \in \mathscr{T}$ such that $x \in A \subseteq A^- \subseteq G_{r(\delta)}$. Thus $x_\delta \leqslant \delta\chi_A \leqslant \delta\chi_{A^-} \leqslant r(\delta)\chi_{G_{r(\delta)}} \leqslant U$. Since $(L^X, \omega_L(\mathscr{T}))$ is induced, we have that $\delta\chi_A \in \omega_L(\mathscr{T})$ and $\delta\chi_{A^-} \in (\omega_L(\mathscr{T}))'$. Thus $(\delta\chi_A)^- \leqslant \delta\chi_{A^-}$. Taking $V_\delta = \delta\chi_A$, then $x_\delta \leqslant V_\delta \leqslant (V_\delta)^- \leqslant U$. By the

condition listed in this theorem, $x_\lambda = \vee_{\delta\in\beta^*(\lambda)} x_\delta \leqslant \vee_{\delta\in\beta^*(\lambda)} V_\delta \leqslant (\vee_{\delta\in\beta^*(\lambda)} x_\delta)^- = \vee_{\delta\in\beta^*(\lambda)} V_\delta^- \leqslant U$ . Taking $V = \vee_{\delta\in\beta^*(\lambda)} V_\delta$ , then $x_\lambda \leqslant V \leqslant V^- \leqslant U$ . So $(L^X, \omega_L(\mathscr{T}))$ is a containwise regular space.

From Theorem 2, we know that if $(L^X, \omega_L(\mathscr{T}))$ is a closure preserving space for the family of open sets, then $(L^X, \omega_L(\mathscr{T}))$ is a containwise regular space iff its underlying space $(X, \mathscr{T})$ is a regular space.

**Lemma 1**[2] $(X, \delta)$ is a containwise regular $F$ topological space iff for every $F$ point $x_\lambda$ and open $Q$ -neighborhood $U$ of $x_\lambda$ there exists an open $Q$ -neighborhood $V$ of $x_\lambda$ such that $V^- \subseteq U$ .

**Lemma 2** $(X, \delta)$ is a containwise regular $F$ topological space iff for every $F$ point $x_\lambda$ and every $P \in \eta^-(x_\lambda)$ there exists $Q \in \eta^-(x_\lambda)$ such that $P \leqslant Q^\circ$ .

**Proof** It is easy to prove from the fact that $P \in \eta^-(x_\lambda)$ iff $P'$ is an open $Q$ -neighborhood of $x_\lambda$ and Lemma 1.

**Theorem 3** Let $(X, \delta)$ be an induced space generated by $(X, \mathscr{T})$ . Then $(X, \delta)$ is a containwise regular space iff its underlying space $(X, \mathscr{T})$ is a regular space.

**Proof** By Lemma 2 and Theorem 4. 8 in [3], the proof is obvious.

Theorem 3 shows us that the containwise regularity is a good extension in the sense of Lowen.

## References

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