On the fuzzy continuity of convex fuzzy mapping

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Abstract: In the sense of the convex fuzzy mapping given by Nanda [1], we define the lower semi continuous fuzzy mapping and the continuous fuzzy mapping in this paper. And on the n - dimensional Euclidean space, we obtain an important result which the convex fuzzy mapping is continuous fuzzy mapping on its relative interior.

Keywords: Convex fuzzy mappings; Lower semicontinuous fuzzy mappings; Continuous fuzzy mappings (fuzzy continuation); Relative interior; Epigraph.

1 Introduction

The concept of convex fuzzy sets were originally introduced by L. A. Zadeh [2]. Subsequently a lot of scholars did a great deal of work at the aspects of their theories and applications. Some properties of convex fuzzy sets were studied and given by Brown [3], Katsaras and Liu [4], Lowen [5]. The concept of the convex fuzzy mapping has first been introduced and some results including some applications to nondifferentiable optimization have been investigated by Nanda [1]. But the theories of the continuity question on convex fuzzy mapping have not been discussed up to now. In this paper, we will study them. For simplicity, we consider only the convex fuzzy mapping defined on the Euclidean space R^n, But it is not difficult to generalize the results obtained here to the case that convex fuzzy mapping are defined in a linear space over the real or complex field.

2 Preliminaries

Throughout this paper, let R be the set of all real numbers.

Let $I_R = \{\bar{a} = [a^-, a^+] \mid a^- \leqslant a^+, a^-, a^+ \in R\}$. The elements in the set I_R are called interval numbers. On the I_R, we make following definitions.

For every $\bar{a}$, $\bar{b} \in I_R$ *and* $\bar{a} = [a^-, a^+]$, $\bar{b} = [b^- b^+]$,

$\bar{a} + \bar{b} = [a^- + b^-, a^+ + b^+]$, $k\bar{a} = [ka^-, ka^+]$, Whenever $k \geqslant 0$

$\bar{a} - \bar{b} = [(a^- - b^-) \wedge (a^+ - b^+), (a^- - b^-) \vee (a^+ - b^+)]$,

$\bar{a} \leqslant \bar{b}$ iff $a^- \leqslant b^-$, $a^+ \leqslant b^+$, $\bar{a} = \bar{b}$ iff $a^- = b^-$, $a^+ = b^+$,

$\bar{a} < \bar{b}$ iff $\bar{a} \leqslant \bar{b}$ *and* $\bar{a} \neq \bar{b}$,

Evidently, we obtain ① $(\bar{a} + \bar{b}) - \bar{b} = \bar{a}$, $\bar{a} - \bar{a} = \bar{0} = [0,0]$

② $k(\bar{a} \pm \bar{b}) = k\bar{a} \pm k\bar{b}$, Whenever $k \geqslant 0$

Let $\rho(\bar{a}, \bar{b}) = \max\{|a^- - b^-|, |a^+ - b^+|\}$, Clearly, (I_R, ρ) is a (Hausdorff) metric space.

For a sequence of interval numbers $\{\bar{a}_n\} \subset I_R$, We say that $\bar{a}_n \to \bar{a}(n \to \infty)$ if there exists a $\bar{a} \in I_R$ such that $\rho(\bar{a}_n, \bar{a}) \to 0(n \to \infty)$. Obviously we say $\bar{a}_n \to \bar{a}(n \to \infty)$ or $\lim_{n\to\infty} \bar{a}_n = \bar{a}$ iff $a_n^- \to a^-$ *and* $a_n^+ \to a^+(n \to \infty)$.

Let A: $R \to [0,1]$. If there exists an $x_0 \in R$ such that $A(x_0) = 1$ and for any $\lambda \in (0,1]$, the cut-set $A_\lambda = \{x \in R \mid A(x) \geqslant \lambda\} \in I_R$. Then we call A a fuzzy number on R. simply write as $A_\lambda = [A_\lambda^-, A_\lambda^-]$. We denote all fuzzy numbers on R as $F(R)$.

For $A, B \in F(R), k \geqslant 0$, we define

$$(kA)(x) = \begin{cases} A(\frac{x}{k}) & & \text{if } k \neq 0 \\ 1 & \text{for } x = 0 & \text{if } k = 0 \\ 0 & \text{for } x \neq 0 & \text{if } k = 0 \end{cases}$$

$A \leqslant B$ iff $A_\lambda \leqslant B_\lambda$ for any $\lambda \in (0,1]$;

$A < B$ iff $A_\lambda \leqslant B_\lambda$ and $A_\lambda \neq B_\lambda$ for any $\lambda \in (0,1]$;

$A \pm B = C$ *where* $C_\lambda = A_\lambda \pm B_\lambda$ for any $\lambda \in (0,1]$.

Evidently, we have $(A + B) - B = A$ and $A - A = \tilde{0}$, At the same time, $A \leqslant B$ iff $A - B \leqslant \tilde{0}$ holds. where $\tilde{0}(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$

By above definitions, we are not difficult to get the following propositions.

Proposition 2.1 Let $A, B \in F(R)$, Then the following conclusions hold.

(1) For any $\lambda \in (0,1]$, if $k \geqslant 0$, then $(kA)_\lambda = kA_\lambda$.

(2) If $A \leqslant B$ and $k \geqslant 0$, then $kA \leqslant kB$.

(3) If $k \geqslant 0$, then $k(A \pm B) = kA \pm kB$.

(4) If $k, l \geqslant 0$, then $(kl)(A) = k(lA)$.

Finally, define a mapping $\tilde{\rho}: F(R) \times F(R) \to [0, \infty)$, and by the rule $\tilde{\rho}(A, B) = \sup_{\lambda \in (0,1]} \rho(A_\lambda, B_\lambda)$. It is straightforward to see that $(F(R), \leqslant)$ constitutes a partial ordered set, and $\tilde{\rho}$ is a metric in $F(R)$.

For a sequence of fuzzy numbers $\{A_n\} \subset F(R), A \in F(R)$. We say that $\{A_n\}$ is convergent to A iff $\tilde{\rho}(A_n, A) \to 0$ $(n \to \infty)$. Simply write as $\lim_{n \to \infty} A_n = A$ *or* $A_n \to A$ $(n \to \infty)$. Evidently, $A_n \to A (n \to \infty)$ iff $(A_n)_\lambda^- \to A_\lambda^-$ and $(A_n)_\lambda^+ \to A_\lambda^+ (n \to \infty)$ for any $\lambda \in [0,1]$.

Proposition 2.2 Let two sequence of fuzzy numbers $\{A_n\}$, $\{B_n\} \subset F(R), A, B \in F(R)$. and $\lim_{n \to \infty} A_n = A, \lim_{n \to \infty} B_n = B$. *If* $A_n \leqslant B_n$, $n = 1, 2, 3, \cdots$. Then $A \leqslant B$.

3 Continuity of the convex fuzzy mapping

This section is the centre of this paper. We first define the concept of the lower semi continuous fuzzy mapping and the continuous fuzzy mapping. Furthermore, we conclude that the convex fuzzy mapping defined on the Euclidean space R^n is the fuzzy continuous.

Let R^n always denote the n - dimensional Euclidean space. The subset $M \subset R^n$ is called an affine set implies that for all $x, y \in M$, $\forall \lambda \in R, (1 - \lambda)x + \lambda y \in M$ holds.

Let $S \subset R^n$. We call the smallest affine set containing S an affine hull of S. Write as aff S. i.e., aff $S = \bigcap \{M \mid M$ are affine sets. and $S \subset M \subset R^n\}$.

Definition 3.1[7] Let $M \subset R^n, x \in M$. x is called a relative inner point of M. if there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \cap \text{aff } M \subset M$. Where $B(x, \varepsilon)$ is an open sphere on R^n. The set constituted by all relative inner points of M is called the relative interior of M. Write as ri$M$. i.e., ri$M = \{x \in M \mid$ there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \cap \text{aff } M \subset M\}$.

Lemma 3.1[7] Let M be a nonempty convex subset in R^n. Then $y \in \text{ri}M$ iff for any $x \in M$, there exists a $\alpha > 1$ such that $(1 - \alpha)x + \alpha y \in M$.

Definition 3.2 Let $\Omega \subset R^n$ be a convex set, and $f : \Omega \to F(R)$ be a fuzzy mapping. Then f is said to be a convex fuzzy mapping if for any $x, y \in \Omega$, and $\lambda \in [0, 1]$ implies $f((1 - \lambda)x + \lambda y) \leqslant (1 - \lambda)f(x) + \lambda f(y)$.

Lemma 3.2[1] Let $\Omega \subset R^n$ be a convex set, and $f: \Omega \to F(R)$ be a fuzzy mapping. Then f is a convex fuzzy mapping if and only if its epigraph epi$f = \{(x, \mu) \mid x \in \Omega, \mu \in F(R), f(x) \leqslant \mu\}$ is a convex set in $\Omega \times F(R)$.

Definition 3.3 Let closed set $U \subset R^n$, $x_0 \in U$, mapping $f: U \to F(R)$. If for any $\{x_k\} \subset U$, $\lim_{k\to\infty} x_k = x_0$ and $\lim_{k\to\infty} f(x_k)$ exists implies $\lim_{k\to\infty} f(x_k) \geqslant f(x_0)$ (*or* $\lim_{k\to\infty} f(x_k) \leqslant f(x_0)$). Then f is called a lower (or upper) semicontinuous fuzzy mapping at point x_0 . If f is both an upper semicontinuous fuzzy mapping and a lower semicontinuous fuzzy mapping at point x_0 . Then f is called a continuous fuzzy mapping at point x_0.

Similarly, if f is a continuous fuzzy mapping at arbitrary a point in U. Then f is called a continuous fuzzy mapping on U.

Theorem 3.1.Let closed set $U \subset R^n$, $x_0 \in U$, $f: U \to F(R)$. Then f is lower semicontinuous on U iff its epigraph epif is a closed set on $U \times F(R)$.

Proof. Necessity. Taking any $(x_k, \mu_k) \in \text{epi}f (k = 1,2,3,\cdots)$, which satisfies $\lim_{k\to\infty}(x_k, \mu_k) = (x_0, \mu_0)$. Then we have $\{x_k\} \subset U$, $\{\mu_k\} \subset F(R)$, $f(x_k) \leqslant \mu_k (k = 1,2,3,\cdots)$, *and* $\lim_{k\to\infty} x_k = x_0$, $\lim_{k\to\infty} \mu_k = \mu_0 \in F(R)$.

Due to f is lower semicontinuous. we view that $\lim_{k\to\infty} f(x_k)$ exists and $\lim_{k\to\infty} f(x_k) \geqslant f(x_0)$. In addition, according to the definition of convergence of the sequences of fuzzy numbers and proposition 2.2, by $f(x_k) \leqslant \mu_k$, we get $\lim_{k\to\infty} f(x_k) \leqslant \lim_{k\to\infty} \mu_k = \mu_0$. consequently $f(x_0) \leqslant \mu_0$, *i.e.*, $(x_0, \mu_0) \in \text{epi}f$. Which means that epi$f$ is a closed set.

Sufficiency. Taking any $x_0 \in U$ and $\{x_k\} \subset U$ such that $\lim_{k\to\infty} x_k = x_0$.

Let $\lim_{k\to\infty} f(x_k) = \mu$. Then we have $(x_k, f(x_k)) \in \text{epi}f$. Since epi$f$ is a closed set on $U \times F(R)$, it follows that $\lim_{k\to\infty}(x_k, f(x_k)) = (x_0, \mu) \in \text{epi}f$.

Hence $f(x_0) \leqslant \mu = \lim_{k\to\infty} f(x_k)$. i.e., f is lower semicontinuous at point x_0.

From point x_0 is arbitrary, we obtain that f is lower semicontinuous on U .

Theorem3.2. Let closed set $U \subset R^n$, $x_0 \in U$, mapping $f: U \to F(R)$.Then f is lower semicontinuous on U iff for any $\eta \in F(R)$, the level set $S_\eta(f) = \{x \in U \mid f(x) \leqslant \eta\}$ is a closed set.

Proof. Necessity. According to the proof of theorem 3.1, it follows that f is lower semicontinuous at point $x_0 \in U$. Which is equivalent to for every $\{x_k\} \subset U$, $\{\mu_k\} \subset F(R)$, such that $\lim_{k\to\infty} x_k = x_0$, $\lim_{k\to\infty} \mu_k = \mu$ and $f(x_k) \leqslant \mu_k$, $(k = 1,2,3,\cdots)$ implies $f(x_0) \leqslant \mu$. Therefore, let $\eta = \mu = \mu_k$. Then we get immediately that $S_\eta(f)$ is a closed set.

Sufficiency. For each $x_0 \in U$, $\{x_k\} \subset U$ such that $\lim_{k\to\infty} x_k = x_0$, $\lim_{k\to\infty} f(x_k) = \mu_0$.

clearly, for any $\eta > \mu_0$, where $\eta, \mu_0 \in F(R)$, From the definition of convergence of the sequences of fuzzy numbers, it shows that $x_k \in S_\eta(f)$ whenever k is large enough.

Because $S_\eta(f)$ is a closed set on U , we have $x_0 \in S_\eta(f)$ for all $\eta > \mu_0$.

Due to sequence $\{x_k\}$ is arbitrary , it follows that $f(x_0) \leqslant \mu_0 = \lim_{k \to \infty} f(x_k)$.

Consequently , f is lower semicontinuous on U .

Theorem 3.3 Let $\Omega \subset R^n$ be a convex nonempty subset. Let $f: \Omega \to F(R)$ be a convex fuzzy mapping. Then $\mathrm{ri}(\mathrm{epi} f) = \{(x, \mu) \mid x \in \mathrm{ri}\Omega, \mu \in F(R), f(x) < \mu\}$.

Proof. Let $M = \{(x, \mu) \mid x \in \mathrm{ri}\Omega, \mu \in F(R), f(x) < \mu\}$. Then $\mathrm{ri}(\mathrm{epi} f) \subset M$ is obvious. We need only prove $M \subset \mathrm{ri}(\mathrm{epi} f)$

Let $(x_0, u_0) \in M$. *i.e.*, $x_0 \in \mathrm{ri}\Omega$, *and* $f(x_0) < \mu_0 \in F(R)$.

For any $(x, \mu) \in \mathrm{epi} f$. We have $f(x) \leqslant \mu$.

From Lemma 3.1[7], we know that there exists *a* $\lambda_0 > 1$ such that $y = (1 - \lambda_0)x + \lambda_0 x_0 \in \Omega$. Now , if we can find a $\lambda > 1$ such that $(1 - \lambda)(x, \mu) + \lambda(x_0, \mu_0) \in \mathrm{epi} f$ *i.e.*, $z = (1 - \lambda)x + \lambda x_0 \in \Omega$. *and* $f(z) \leqslant (1 - \lambda)\mu + \lambda\mu_0$. By lemma 3.1 [7], we have $(x_0, \mu_0) \in \mathrm{ri}(\mathrm{epi} f)$. Then the proof is completed.

Thus, on the one hand , Let $x(t) = (1 - t)y + tx_0$.

then $x(t) = (1 - t)(1 - \lambda_0)x + ((1 - t)\lambda_0 + t)x_0$

$= (1 - \varphi(t))x + \varphi(t)x_0$.

Where $\varphi(t) = (1 - t)\lambda_0 + t$, *and* $1 < \varphi(t) < \lambda_0$.

Let $\mu_\varepsilon = \mu_0 - f(x_0)$. Since f is a convex fuzzy mapping, we obtain

$f(x(t)) \leqslant (1 - t)f(y) + tf(x_0)$ …… (1)

On the other hand, x_0 can be denoted as $x_0 = (1 - \frac{1}{\lambda_0})x + \frac{1}{\lambda_0}y$.

It shows that $f(x_0) \leqslant (1 - \frac{1}{\lambda_0})f(x) + \frac{1}{\lambda_0}f(y)$. By the operation properties of fuzzy numbers in proposition 2.1, we can get that

$f(y) \geqslant (1 - \lambda_0)f(x) + \lambda_0 f(x_0)$……(2)

Combining (1) and (2) , we have $f(x(t)) \leqslant (1 - \varphi(t))\mu + \varphi(t)\mu_0 - \mu_\varepsilon t + (1 - t)(f(y) + (\lambda_0 - 1)\mu - \lambda_0\mu_0)$.

We choose a proper $t \in (0,1)$ such that $(f(y) + (\lambda_0 - 1)\mu - \lambda_0\mu_0) < \frac{t}{1 - t}\mu_\varepsilon$. let $\lambda = \varphi(t)$. Then $\lambda > 1$. From the operation properties of fuzzy numbers and proposition 2.1, we have $f((1 - \lambda)x + \lambda x_0) \leqslant (1 - \lambda)\mu + \lambda\mu_0$.

That is $(1 - \lambda)(x, \mu) + \lambda(x_0, \mu_0) \in \mathrm{epi} f$. thus, theorem is completed.

Now , we are going to verify an important result of this paper applying to above theorems proved.

Theorem 3.4. Let $\Omega \subset R^n$ be a convex nonempty subset . Let $f: \Omega \to F(R)$ be a convex fuzzy mapping . Then f is fuzzy continuous on the relative interior riΩ .

Proof. First. according to theorem 3.1 and theorem 3.3, we are easy to know that f is lower semicontinuous at arbitrary point on riΩ.

Second. We need only prove that f is upper semicontinuous on riΩ. Thus, for each $x_0 \in$ riΩ. Let $\{x_k\} \subset$ riΩ $(k = 1,2,3,\cdots)$ such that $\lim_{k\to\infty} x_k = x_0$, *and* $\lim_{k\to\infty} f(x_k) = \mu \in F(R)$. Now we will verify that $\mu \leqslant f(x_0)$.

Indeed, otherwise, if $f(x_0) < \mu$. By theorem 3.3, implies $(x_0, \mu) \in$ ri(epi$f$). In the meantime, we have $\lim_{k\to\infty}(x_k, f(x_k)) = (x_0, \mu)$. Whenever k is large enough, from the definitions of convergence of the sequences of fuzzy numbers and interval numbers , we get $(x_k, f(x_k)) \in$ ri(epi$f$). Consequently, it shows that $f(x_k) < f(x_k)$. This is impossible. Hence f is upper semicontinuous at point x_0. Which means that f is fuzzy continuous at point x_0.

Since x_0 is arbitrary. f is fuzzy continuous on riΩ.

References.

[1] S. Nanda and K. Kar , Convex fuzzy mapping , Fuzzy Sets and Systems 48 (1992) 129 - 132.
[2] L. A. Zadeh. Fuzzy sets, Information and Control, 8 (1965) 338 - 353.
[3] Brown, J. G., A note on fuzzy sets , Information and control 18 (1971) 32 - 39.
[4] Katsaras, A. K. and Liu , D. B., Fuzzy vector spaces and fuzzy topological vector spaces ibid. 58 (1977) 135 - 146.
[5] R. Lowen. Convex fuzzy sets , Fuzzy Sets and Systems, 3 (1980) 291 - 310.
[6] S. Nanda, Fuzzy liner spaces over valued fields , Fuzzy Sets and Systems 42 (1991) 351 - 354.
[7] Feng Dexing , The Basis of Convex Analysis. Scientific Publishing House . (China) 1995
[8] Yu Yandong, On the convex fuzzy sets(I), Fuzzy Math. (China), No.2 (1984).
[9] Xinmin Yang , A note on convex fuzzy sets , Fuzzy Sets and Systems 53 (1993) 117 - 118.
[10] Wang Guijun and Li Xiaoping, On the convergence of the fuzzy valued functional. The Journal of Fuzzy Mathematics, Vol.5, 2(1997), 431—438.