The Axiomatic System of Fuzzy Sets Theory

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Abstract:

Based on the elementary fuzzy point, a definition of the L-fuzzy set is presented. Furthermore, an axiomatic system of the L-fuzzy set theory is established.

Keywords:

Elementary fuzzy point, L-fuzzy set, membership function, L-fuzzy point.

1. Introduction

Zadeh gave the concept of fuzzy set by means of membership function in [1]. Fuzzy set was usually equated with its membership function in general references of fuzzy sets theory. The concepts of ensembles floues and nested sets were introduced in [2] [3] [4]. Furthermore, the expression theorem and another concept of fuzzy set were given. The concepts above of fuzzy set are all based on the mappings and belong to nonpoint types. In [5], the reasonable definition of elementary fuzzy point and axiomatic definition of fuzzy set were given. In this paper, the axiomatic definition of fuzzy set is given from another side. And the corresponding axiomatic system is established. Therefore, the close logical foundation are settled for fuzzy sets theory.

2. Axiomatic definition of LF set

Definition 2.1. Let X be nonempty set, $(L, \leqslant, \vee, \wedge)$ be a complete lattice. $L_0 = L - \{0\}$, $X(L) \triangleq X \times L_0 = \{(x, \lambda) : x \in X, \lambda \in L_0\}$. The element (x, λ) in $X(L)$ is called elementary fuzzy point of X. where, x is called support point of (x, λ), λ is called value (or level) of (x, λ).

The order relation in $X(L)$ is defined as follows:

$(x, \lambda) \leqslant (y, \mu) \Longleftrightarrow x = y, \lambda \leqslant \mu$, for any $(x, \lambda), (y, \mu) \in X(L)$.

Proposition 2.1. $(X(L), <)$ is an nontotal order.

Definition 2.2. Set A is callted L-fuzzy set of X (or LF set simply) if $A \subset X(L)$ or $A=\phi$ or A satisfies the following

(1) $(x, \lambda) \in A \Rightarrow (x, \mu) \in A$ for any $\mu \in L_0$ and $\mu > \lambda$;

(2) $\{(x, \lambda_t) : t \in T\} \subset A$, and $\wedge_{t\in T} \lambda_t = \lambda \Rightarrow (x, \lambda) \in A$.

especially, A is the fuzzy set of X if $L=[0,1]$. And A is the classical set if $L=\{0,1\}$、$F_L(X)$、$F(X)$ and $P(X)$ denotes the all LF、fuzzy and classical set on X respectively.

Definition 2.3. Let $A, B \in F_L(X)$

(1) A is called LF subset of B, if for any $(x, \lambda) \in A \Rightarrow (x, \lambda) \in B$. We write $A \subset B$ or $B \supset A$;

(2) A is equal to B if $A \subset B$ and $B \supset A$, we write $A=B$.

Definition 2.4. Let $A \in F_L(X)$. The mapping $A(\cdot): X \to L$ is called membership function of A. Where, $A(X) = \wedge\{\lambda : (x, \lambda) \in A\}$ (we stipulate $\wedge\phi = 1$).

From above definitions, we get easily.

Proposition 2.2. Let $A, B \in F_L(X)$, then

(1) $(x, A(X)) \in A$ if $A(x) \neq 0$;

(2) $A = \{(x, \lambda) \in X(L) : A(X) < \lambda\}$;

(3) $A \subset B \Longleftrightarrow B(X) < A(X)$ for any $x \in X$;

(4) $A = B \Longleftrightarrow A(X) = B(X)$ for any $x \in X$.

Proposition 2.3. $(F_L(X), \subset)$ is a complete lattice.

Proof. It is clear that $(F_L(X), \subset)$ is an nontotal order. For any $A_t \in F_L(X)$, $t \in T$, Let

$A^t = \{(x, \lambda) \in X(L) : \wedge_{t\in T} A_t(x) < \lambda\}$

$A_t = \{(x, \lambda) \in X(L) : \vee_{t\in T} A_t(x) < \lambda\}$

$A^t, A_t \in F_L(X)$ and $A^t = \sup_{t\in T} A_t$, $A_t = \inf_{t\in T} A_t$ can be proved easily. $F_L(X)$ have the maximal element $X(L)$ and the minimal element ϕ. Therefore, $(F_L(X), \subset)$ is a complete lattice.

Definition 2.5. Let $\{A_t; t\in T\}\subset F_L(X)$. We define

$\bigcup_{t\in T}A_t=\{(x,\lambda)\in X(L); \wedge_{t\in T}A_t(X)<\lambda\}$

$\bigcap_{t\in T}A_t=\{(x,\lambda)\in X(L); \vee_{t\in T}A_t(X)<\lambda\}$

Obviously, we have

Proposition 2.4. Let $\{A_t; t\in T\}\subset F_L(X)$. Then, for any $x\in X$,

$(\bigcup_{t\in T}A_t)(x)=\wedge_{t\in T}A_t(x)$, $(\bigcap_{t\in T}A_t)(x)=\vee_{t\in T}A_t(x)$

$L^X=\{\mu : X\to L\}$ denotes the all mappings from X onto L. If we define an nontotal order "$<'$" pointwisely and the following operations in L^X

$(\vee'_{t\in T}\mu_t)(x)=\wedge_{t\in T}\mu_t(x)$, $(\wedge'_{t\in T}\mu_t)(x)=\vee_{t\in T}\mu_t(x)$

where $\mu_t\in L^X$, $t\in T$, $x\in X$. Then $(L^X, <', \vee', \wedge')$ is a complete lattice.

Proposition 2.5. $(F_L(X), \subset, \bigcup, \bigcap)$ and $(L^X, <', \vee', \wedge')$ are isomorphic complete lattice.

Proof. First, $(F_L(X), \subset, \bigcup, \bigcap)$ and $(L^X, <', \vee', \wedge')$ are all complete lattice. Let mapping $f: F_L(X)\to L^X, A\to f(A)$, where, $f(A)(x)=\wedge\{\lambda : (x,\lambda)\in A\}=A(x)$ for any $x\in X$.

(1) f is injection.

Let $A, B\in F_L(X)$ and $f(A)=f(B)$, then $A(x)=B(x)$ for any $x\in X$. From Proposition 2.2(4), we get $A=B$.

(2) f is surjection.

Let $\widetilde{A}\in L^X$, $\widetilde{A}: X\to L$, $A=\{(x,\lambda)\in X(L) : \widetilde{A}(x)<\lambda\}\subset X(L)$.

It is obvious that $A\in F_L(X)$ and $f(A)(x)=\wedge\{\lambda : (x,\lambda)\in A\}=A(x)$. Therefore $f(A)=\widetilde{A}$.

(3) f is homomorphism between complete lattices.

Let $\{A_t : t\in T\}\subset F_L(X)$, then for each $x\in X$,

$$f(\bigcup_{t\in T}A_t)(x)=\wedge\{\lambda; (x,\lambda)\in \bigcup_{t\in T}A_t\}=(\bigcup_{t\in T}A_t)(x)$$
$$=\wedge_{t\in T}A_t(x)=\wedge_{t\in T}f(A_t)(x)=(\vee'_{t\in T}f(A_t))(x)$$

$$f(\bigcap_{t\in T}A_t)(x)=\wedge\{\lambda : (x,\lambda)\in \bigcap_{t\in T}A_t\}=(\bigcap_{t\in T}A_t)(x)$$
$$=\vee_{t\in T}A_t(x)=\vee_{t\in T}f(A_t)(x)=(\wedge'_{t\in T}f(A_t))(x)$$

From above, f remains operations. Therefore, this proposition holds.

Definition 2.6. Let $x \in X$, $\lambda \in L_0$. The set $x_\lambda = \{(x, \delta);\ \delta \in L_0 \text{ and } \delta > \lambda\}$ is called L-fuzzy point(or LF point simplely). Where x is called support point and λ is called value. X_L denotes the all LF point.
Obviously, we have

Proposition 2.6. $(X_L, \subset)$ and $(X(L), <)$ are isomorphic nontotal order sets.

3.Decomposition of LF set

Definition 3.1. (1) Let $A \in F_L(X)$, $\lambda \in L$. The set

$$A_\lambda = \begin{cases} \{x : (x, \lambda) \in A, & \lambda \neq 0 \\ \phi, & \lambda = 0 \end{cases}$$

is called λ-cut set of A・ $A_{\lambda\cdot} = \bigcup_{\mu<\lambda} A_\mu$ is called strong λ-cut set of A.

(2) Let $B \in P(X)$, $\lambda \in L$. The set

$$\lambda B = \begin{cases} \{(x, \mu) \in X(L) : x \in B, \lambda < \mu\}, & \lambda \neq 0 \text{ and } B \neq \phi \\ \phi, & \text{otherwise} \end{cases}$$

is called the product of λ and B.

Obviously, $\lambda B \in F_L(X)$, and $(\lambda B)(x) = \begin{cases} \lambda, & x \in B \\ 1, & x \notin B \end{cases}$ for any $x \in X$.

Proposition 3.1. Let $A \in F_L(X)$. Then

(1) For each $\lambda \in L_0$, we have $x \in A_\lambda$ if and only if $A(x) < \lambda$;

(2) For each $\lambda \in L_0$, we have $x \in A_{\lambda\cdot}$ if and only if $A(x) < \lambda$;

(3) (Decomposition theorem I) $A = \bigcup_{\lambda\in L} \lambda A_\lambda$

Proof. (1) For each $\lambda \in L_0$, $x \in A_\lambda \Longleftrightarrow (x, \lambda) \in A \Longleftrightarrow A(X) < \lambda$.

(2) First, $x \in A_{\lambda\cdot} \Rightarrow x \in \bigcup_{\mu<\lambda} A_\mu \Rightarrow$ there exists $\mu < \lambda$, such that $x \in A_\mu \Rightarrow A(x) < \mu < \lambda$;
On the other hand, $A(x) < \lambda \Rightarrow$ we take $\mu_0 = A(x) < \lambda$, then $x \in A_{\mu_0} \Rightarrow x \in \bigcup_{\mu<\lambda} A_\mu = A_{\lambda\cdot}$

(3) For any $x \in X$, we have

$$(\bigcup_{\lambda\in L} \lambda A_\lambda)(x) = \bigwedge_{\lambda\in L} (\lambda A_\lambda)(x) = \bigwedge\{\lambda : \lambda \in L,\ x \in A_\lambda\}$$
$$= \bigwedge\{\lambda : \lambda \in L,\ A(x) < \lambda\} = A(x)$$

Proposition 3.2. Let L be a dense complete lattice, then

(1) $A_\lambda = \bigcap_{\mu > \lambda} A_{\mu .}$;

(2) (Decomposition theorem II) $A = \bigcup_{\lambda \in L} \lambda A_{\lambda .}$;

Proof. (1) For each $\lambda \in L_n$. First, $x \in A_\lambda \Rightarrow A(x) \leqslant \lambda < \mu$ for every $\mu > \lambda \Rightarrow x \in A_{\mu .}$ for every $\mu > \lambda \Rightarrow x \in \bigcap_{\mu > \lambda} A_{\mu .}$; On the other hand, $x \in \bigcap_{\mu > \lambda} A_{\mu .} \Rightarrow x \in A_{\mu .}$ for any $\mu > \lambda \Rightarrow A(x) < \mu$ for any $\mu > \lambda \Rightarrow A(x) \leqslant \bigwedge_{\mu > \lambda} \mu = \lambda \Rightarrow x \in A_\lambda$

(2) For each $x \in X$, we have

$$(\bigcup_{\lambda \in L} \lambda A_{\lambda .})(x) = \bigwedge_{\lambda \in L} (\lambda A_{\lambda .})(x) = \bigwedge \{ \lambda : \lambda \in L,\ x \in A_{\lambda .} \}$$
$$= \bigwedge \{ \lambda : \lambda \in L,\ A(x) < \lambda \} = A(x).$$

From proposition 3.1. and 3.2., we get

Proposition 3.3. (Decomposition theorem III) Let L be a dense complete lattice, $A \in F_L(X)$. If the mapping $H_n : L \to P(X)$ Satisfies $A_{\lambda .} \subset H_n(\lambda) \subset A_\lambda$ for any $\lambda \in L$, then $A = \bigcup_{\lambda \in L} \lambda H_n(\lambda)$ and $A_\lambda = \bigcap_{\alpha > \lambda} H_n(\alpha)$, $A_{\lambda .} = \bigcup_{\alpha < \lambda} H_n(\alpha)$.

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