NEW EQUILIBRIUM EXISTENCE THEOREMS FOR ABSTRACT FUZZY ECONOMIES

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Abstract. In this paper, we prove some new existence theorems of equilibrium and maximal element for abstract fuzzy economies with uncountable number of agents and qualitative fuzzy games.

Keywords: Fixed point, fuzzy mapping, equilibrium point, abstract fuzzy economy, qualitative fuzzy game.

1. INTRODUCTION

In 1950, Nash [14] first proved the existence of equilibrium for games where the player's preferences are representable by continuous quasiconcave utilities and the strategy sets are simplexes. Next Debreu [7] and Arrow-Debreu [1] proved the existence of social equilibrium and Walrasian equilibrium respectively. In the last twenty years, the classical results has been generalized in many directions. For further details we refer to [2, 4, 5, 6, 8–22, 24, 25] and the references therein.

On the other hand, in a recent book [4], Billot studied the equilibrium points of fuzzy games and fuzzy economic equilibrium, and proved the existence of a fuzzy general equilibrium. As Zimmermann pointed out in the preface of [4], Billot's work is very interesting and it can be hoped that many economic theorists start from his work and advanced economic theory along the lines indicated.

Recently, the author [11, 12] first introduced the concepts of abstract fuzzy economies and generalized abstract fuzzy economies, and studied the equilibrium existence theorems for abstract fuzzy economies and generalized abstract fuzzy economies.

Motivated and inspired by the recent research works [10–12], in this paper, we prove a new equilibrium existence theorem for abstract fuzzy economies with uncountable number of agents with fuzzy constraint correspondences and fuzzy preference correspondence. We also give a new existence theorem of maximal element for qualitative fuzzy games with uncountable number of players with fuzzy preference correspondence.

2. PRELIMINARIES

We firt give some concepts and notations.

A topological space is called to be acyclic if all of its reduced Cech homology groups over rationals vanish. It is well know that any contractible space is acyclic, and so any nonempty convex or star-shaped set is acyclic.

Let X be a topological space and $\mathcal{B}(X)$ the family of all nonempty finite subsets of X. Let $\{\Gamma_A\}$ be a family of nonempty contractible subsets of X indexed by $A \in \mathcal{B}(X)$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B$. The pair $(X, \{\Gamma_A\})$ is called a H-space. Given a H-space $(X, \{\Gamma_A\})$, a nonempty subset D of X is called to be

(1) H-convex if $\Gamma_A \subset D$ for each $A \in \mathcal{B}(D)$;

(2) weakly H-convex if $\Gamma_A \cap D$ is nonempty contractible for each $A \in \mathcal{B}(D)$;

(3) H-compact in X if for each $A \in \mathcal{B}(X)$, there exists a compact weakly H-convex subset D_A of X such that $D \cup A \subset D_A$.

A H-space $(X, \{\Gamma_A\})$ is called to be locally convex H-space if X is a uniform topological space and there exists a basis $\{V_i : i \in I\}$ for the uniform structure such that for each $i \in I$ and each $x \in X$, $V_i[x] = \{y \in X : (x, y) \in V_i\}$ is H-convex.

Let A be a subset of a topological space. We shall denote by 2^A and $\overline{A}$ the family of all subsets of A and the closure of A in X respectively.

Let X, Y be two topological spaces and $T : X \to 2^Y$ be a multivalued mapping. T is said to be upper semicontinuous (respectively, almost upper semicontinuous) if for any $x \in X$ and any open set V in Y with $T(x) \subset V$, there exists an open neighborhood U of x in X such that $T(z) \subset V$ (respectively, $T(z) \subset \overline{V}$) for all $z \in U$. Obviously, an upper semicontinuous multivalued mapping is almost upper semicontinuous ([15, 20]). It is clear that T is upper semicontinuous if and only if for any open set M in Y, the set $\{x \in X : T(x) \subset M\}$ is open in X.

Let M and N be two Hausdorff topological spaces and $X \subset M$, $Y \subset N$ be two nonempty convex subsets. Throughout this paper we always denote by $\mathcal{F}(X)(\mathcal{F}(Y))$ the collection of all fuzzy sets on $X(Y)$. A mapping from X into $\mathcal{F}(Y)(\mathcal{F}(X))$ is called a fuzzy mapping. If $F : X \to \mathcal{F}(Y)$ is a fuzzy mapping, then for each $x \in X$, $F(x)$ (denote by F_x in the sequel) is a fuzzy set in $\mathcal{F}(Y)$ and $F_x(y)$ is the degree of membership of point y in F_x.

In the sequel, we denote by

$$(A)_q = \{x \in X : A(x) \geq q\}, \quad q \in [0, 1]$$

the q-cut set of $A \in \mathcal{F}(X)$.

An abstract fuzzy economy (or generalized fuzzy game) $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ is defined as a family of order quadruples (X_i, A_i, B_i, P_i), where I is a finite or an infinite set of agents, X_i is a nonempty topological space (a choice set), A_i, $B_i : \prod_{k\in I} X_k \to \mathcal{F}(X_i)$ are fuzzy constraint mappings (fuzzy constraint correspondences), and $P_i : \prod_{k\in I} X_k \to \mathcal{F}(X_i)$ is a fuzzy preference mapping (fuzzy preference correspondence).

An equilibrium for Γ is a point $\hat{x} \in \prod_{k\in I} X_k$ such that for each $i \in I$, $\hat{x}_i \in \overline{(B_{i\hat{x}})_{b_i(\hat{x})}}$ and $(P_{i\hat{x}})_{p_i(\hat{x})} \cap (A_{i\hat{x}})_{a_i(\hat{x})} = \emptyset$, where a_i, b_i, $p_i : \prod_{k\in I} X_k \to (0,1]$. If A_i, B_i, $P_i : \prod_{k\in I} X_k \to 2^{X_i}$ are classical set-valued mappings, then the $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$ is an abstract economy (see [5, 9, 10, 15, 18]).

$\Gamma = (X_i, P_i)_{i\in I}$ is said to be a qualitative fuzzy game if for any $i \in I$, X_i is a strategy set of player i, and $P_i : \prod_{k\in I} X_k \to \mathcal{F}(X_i)$ is a fuzzy preference mapping (fuzzy preference correspondence) of player i.

A maximal element of Γ is a point $\hat{x} \in \prod_{k\in I} X_k$ such that $P_{i\hat{x}}(x) < p_i(\hat{x})$ for all $i \in I$ and $x \in \prod_{k\in I} X_k$, where $p_i : \prod_{k\in I} X_k \to (0,1]$. When $P_i : \prod_{k\in I} X_k \to 2^{X_i}$ is a classical set-valued mapping, then the $\Gamma = (X_i, P_i)_{i\in I}$ is a qualitative game.

LEMMA 2.1[23]. *Let $\{(X_i, \{\Gamma_{A_i}\}) : i \in I\}$ be a family of compact Hausdorff locally convex H-space and $X = \prod_{k\in I} X_k$. If for each $i \in I$, $T_i : X \to 2^{X_i}$ is an upper semicontinuous multivalued mappings with nonempty closed acyclic values, then there exists a point $\hat{x} = \prod_{k\in I} \hat{x}_k \in X$ such that $\hat{x}_i \in T_i(\hat{x})$ for all $i \in I$.*

3. EQUILIBRIUM EXISTENCE THEOREMS

In this section. we give a new equilibrium existence theorem for abstract fuzzy economies and a new maximal element existence theorem for qualitative fuzzy games.

THEOREM 3.1. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$ be an abstract fuzzy economy and a_i, b_i, $p_i : \prod_{k\in I} X_k \to (0,1]$ such that for each $i \in I$, the following conditions satisfied:*

(1) X_i is a Hausdorff locally convex H-space and D_i is a nonempty H-compact subset of X_i;

(2) for all $x \in X = \prod_{i\in I} X_i$, $(P_{ix})_{p_i(x)} \subset D_i$, $(A_{ix})_{a_i(x)} \subset D_i$, $\emptyset \neq (B_{ix})_{b_i(x)} \subset D_i$, and $\overline{(B_{ix})_{b_i(x)}} \cap \overline{(P_{ix})_{p_i(x)}}$ and $\overline{(B_{ix})_{b_i(x)}}$ are acyclic;

(3) the set $W_i = \{x \in X : (A_{ix})_{a_i(x)} \cap (P_{ix})_{p_i(x)} \neq \emptyset\}$ is open in X;

(4) mappings $x \mapsto (B_{ix})_{b_i(x)}$ and $x \mapsto (P_{ix})_{p_i(x)}$ are almost upper semicontinuous;

(5) for each $x \in W_i$, $x_i \notin \overline{(B_{ix})_{b_i(x)}} \cap \overline{(P_{ix})_{p_i(x)}}$.

Then there exists a point $\hat{x} \in D = \prod_{i\in I} D_i$ such that

$$\hat{x}_i \in \overline{(B_{i\hat{x}})_{b_i(\hat{x})}} \text{ and } (P_{i\hat{x}})_{p_i(\hat{x})} \cap (A_{i\hat{x}})_{a_i(\hat{x})} = \emptyset$$

for all $i \in I$.

PROOF. For each $i \in I$, since D_i is H-compact, there exists a compact weakly H-convex subset E_i of X_i such that $D_i \subset E_i$. For each $x \in E_i$ and each open neighborhood G of $\overline{(B_{ix})_{b_i(x)}}$ in E_i, there exists an open neighborhood U of x in X such that

$$(B_{iz})_{b_i(z)} \subset D_i \cap \overline{V} \subset \overline{V}, \ \forall z \in U.$$

Hence

$$\overline{(B_{iz})_{b_i(z)}} \subset \overline{V} \subset G, \ \ \forall z \in U \cap E,$$

where $E = \prod_{k \in I} E_k$. This implies that the mapping $x \mapsto \overline{(B_{ix})_{b_i(x)}}$ is upper semicontinuous from E to 2^{E_i}. Similarly, we know that the mapping $x \mapsto \overline{(P_{ix})_{b_i(x)}}$ is upper semicontinuous from E to 2^{E_i}. Therefore, the mapping $x \mapsto \overline{(B_{ix})_{b_i(x)}} \cap \overline{(P_{ix})_{p_i(x)}}$ is upper semicontinuous by Proposition 3.1.7 and Theorem 3.1.8 in [3].

For each $i \in I$ and $x \in X$, let

$$T_i(x) = \begin{cases} \overline{(B_{ix})_{b_i(x)}} \cap \overline{(P_{ix})_{p_i(x)}}, & \text{if } x \in W_i, \\ \overline{(B_{ix})_{b_i(x)}}, & \text{if } x \notin W_i. \end{cases}$$

Then $T_i(x)$ is a nonempty closed acyclic subset of E_i.

Now we prove that $T_i : E \to 2^{E_i}$ is upper semicontinuous. In fact, suppose that V is any given open subset of E_i containing $T_i(x)$. Let $H_i(x) = \overline{(B_{ix})_{b_i(x)}} \cap \overline{(P_{ix})_{p_i(x)}}$ for all $x \in E$. Then we have

$$\begin{aligned} \{x \in E : T_i(x) \subset V\} &= \{x \in W_i : (A_{ix})_{a_i(x)} \cap (P_{ix})_{p_i(x)} \subset V\} \\ &\quad \cup \{x \in E \backslash W_i : (B_{ix})_{b_i(x)} \subset V\} \\ &\subset \{x \in W_i : H_i(x) \subset V\} \cup \{x \in E : (B_{ix})_{b_i(x)} \subset V\}. \end{aligned}$$

On the other hand, when $x \in W_i$ and $H_i(x) \subset V$, we have $T_i(x) = H_i(x) \subset V$. When $x \in X$ and $(B_{ix})_{b_i(x)} \subset V$, since $H_i(x) \subset (B_{ix})_{b_i(x)}$, we know that $T_i(x) \subset V$ and so

$$\{x \in W_i : H_i(x) \subset V\} \cup \{x \in X : (B_{ix})_{b_i(x)} \subset V\} \subset \{x \in X : T_i(x) \subset V\}.$$

Therefore

$$\begin{aligned} \{x \in E : T_i(x) \subset V\} &= \{x \in W_i : H_i(x) \subset V\} \cup \{x \in E : (B_{ix})_{b_i(x)} \subset V\} \\ &= W_i \cap \{x \in E : H_i(x) \subset V\} \cup \{x \in E : (B_{ix})_{b_i(x)} \subset V\}. \end{aligned}$$

Since H_i and $x \mapsto (B_{ix})_{b_i(x)}$ are upper semicontinuous, the sets $\{x \in E : H_i(x) \subset V\}$ and $\{x \in E : (B_{ix})_{b_i(x)} \subset V\}$ are open. It follows that $\{x \in E : T_i(x) \subset V\}$ is open and so the mapping $T_i : E \to 2^{E_i}$ is upper semicontinuous.

By virtue of Lemma 2.1, there exists a point $\hat{x} \in D = \prod_{i \in I} D_i$ such that $\hat{x}_i \in T_i(\hat{x})$, for all $i \in I$. By the condition (5) we have

$$\hat{x}_i \in \overline{(B_{i\hat{x}})_{b_i(\hat{x})}} \text{ and } (P_{i\hat{x}})_{p_i(\hat{x})} \cap (A_{i\hat{x}})_{a_i(\hat{x})} = \emptyset$$

for all $i \in I$. This completes the proof of Theorem 3.1.

From Theorems 3.1, we can obtain the following result.

THEOREM 3.2. *Let $\Gamma = (X_i, P_i)_{i \in I}$ be an qualitative fuzzy game and $p_i : \prod_{k \in I} X_k \to (0, 1]$ such that for each $i \in I$, the following conditions satisfied:*

(1) X_i is a Hausdorff locally convex H-space and D_i is a nonempty H-compact closed acyclic subset of X_i;

(2) for all $x \in X = \prod_{i \in I} X_i$, $(P_{ix})_{p_i(x)} \subset D_i$ *and* $(P_{ix})_{p_i(x)}$ *is acyclic;*

(3) the set $W_i = \{x \in X : (P_{ix})_{p_i(x)} \neq \emptyset\}$ *is open in* X*;*

(4) mappings $x \mapsto (P_{ix})_{p_i(x)}$ *is almost upper semicontinuous;*

(5) for each $x \in W_i$, $x_i \notin \overline{(P_{ix})_{p_i(x)}}$.

Then there exists a point $\hat{x} \in D = \prod_{i \in I} D_i$ *such that such that* $P_{i\hat{x}}(x) < p_i(\hat{x})$ *for all* $i \in I$, $x \in \prod_{k \in I} X_k$.

REMARK. In Theorems 3.1–3.2, when A_i, B_i, P_i are classical set-valued mappings ($i \in I$), we can obtain some new existence theorems of equilibrium and maximal element for abstract economies and qualitative games respectively.

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