Fuzzy Riemann Integrals Based on the Lower Cut Set

Chen Tuyun* Yu Bin# Yuan Xuehai*

* *Department of Mathematics, Liaoning Normal University, Dalian* 116029, *P. R. China*

Department of Basement, Dalian Navy Academy, Dalian 116018, *P. R. China*

ABSTRACT

In this paper, we build two new kind of fuzzy Riemann integrals based on lower cut set: (1) fuzzy Riemann integral of classical function; (2) fuzzy Riemann integral of fuzzy value function based on concave fuzzy numbers.

Keywords: fuzzy sets, lower cut, concave fuzzy number, fuzzy integrals.

1. Preliminary

Let X be a set and $\mathcal{P}(X)$ be power set of X and $\mathcal{F}(X)$ be a set of fuzzy subsets of X over $[0,1]$. For $\underset{\sim}{A}\in\mathcal{F}(X)$ and $\lambda\in[0,1]$,

$$A^{\lambda}=\{x\in X \mid \underset{\sim}{A}(x)\leqslant\lambda\} \qquad A^{\dot{\lambda}}=\{x\in X \mid \underset{\sim}{A}(x)<\lambda\}$$

are called as λ—lower cut set and λ—strong lower cut set of fuzzy set $\underset{\sim}{A}$ respectively.

Let C be a subset of X, we define λC as a fuzzy subset of X and

$$(\lambda C)(x)=\begin{cases}\lambda, & \text{if } x\in C\\ 1, & \text{if } x\notin C\end{cases}$$

then, we have:

Decomposition theorem (1) $\underset{\sim}{A}=\bigcap_{\lambda\in[0,1]}\lambda A^{\lambda}$, i. e., $\underset{\sim}{A}(x)=\inf\{\lambda\mid\lambda\in[0,1], x\in A^{\lambda}\}$; (2) $\underset{\sim}{A}=\bigcap_{\lambda\in[0,1]}\lambda A^{\dot{\lambda}}$, i. e., $\underset{\sim}{A}(x)=\inf\{\lambda\mid\lambda\in[0,1], x\in A^{\dot{\lambda}}\}$.

Let $H:[0,1]\to\mathcal{P}(X)$, $\lambda\to H(\lambda)$ be a mapping satisfying: $\lambda<\mu\Rightarrow H(\lambda)\subseteq H(\mu)$. We called H as a order set embedding over X. $\mathcal{U}(X)$ is denoted as a set of all order embedding over X.

In $\mathcal{U}(X)$, we define operations $\bigcup, \bigcap, c$ as following:

$$\left(\bigcup_{r\in\Gamma}H_r\right)(\lambda)=\bigcap_{r\in\Gamma}H_r(\lambda)$$

$$(\bigcap_{r\in\Gamma}H_r)(\lambda)=\bigcup_{r\in\Gamma}H_r(\lambda)$$

$$(H^c)(\lambda)=(H(1-\lambda))^c$$

then, we have:

Representation theorem Let $T:\mathscr{U}(X)\to\mathscr{F}(X), H\to T(H)=\bigcap_{\lambda\in[0,1]}\lambda H(\lambda)$, i. e. , $T(H)(x)=\inf\{\lambda|\lambda\in[0,1], x\in H(\lambda)\}$, then T is a homomorphism from $(\mathscr{U}(X),\cup,\cap,c)$ to $(\mathscr{F}(X),\cup,\cap,c)$ and (1) $T(H)_{\lambda}\subseteq H(\lambda)\subseteq T(H)^{\lambda}, \forall\lambda\in[0,1]$; (2) $T(H)^{\lambda}=\bigcap_{\alpha>\lambda}H(\alpha)$, $T(H)_{\lambda}=\bigcup_{\alpha<\lambda}H(\alpha)$

Let $f: X\to Y$ be a function, then we have:

Extension principle Let $f:\mathscr{F}(X)\to\mathscr{F}(Y)$ $\underset{\sim}{A}\to f(\underset{\sim}{A})\triangleq\bigcap_{\lambda\in[0,1]}\lambda f(A^{\lambda})$ then $f(\underset{\sim}{A})(y)=\bigwedge_{f(x)=y}\underset{\sim}{A}(x)$.

2. Fuzzy Riemann integral of classical function

Lemma 1 Let $\underset{\sim}{A}\in\mathscr{F}(X)$, $\underset{\sim}{B}\in\mathscr{F}(Y)$, we define: $\underset{\sim}{A}\otimes\underset{\sim}{B}\triangleq\bigcap_{\lambda\in[0,1]}\lambda(A^{\lambda}\times B^{\lambda})$, then $(\underset{\sim}{A}\otimes\underset{\sim}{B})(x,y)=\underset{\sim}{A}(x)\vee\underset{\sim}{B}(y)$

Lemma 2 Let $f: X\times Y\to Z$ be a function, we define:

$f:\mathscr{F}(X)\times\mathscr{F}(Y)\to\mathscr{F}(Z)$

$(\underset{\sim}{A},\underset{\sim}{B})\to f(\underset{\sim}{A},\underset{\sim}{B})\triangleq f(\underset{\sim}{A}\otimes\underset{\sim}{B})$

then $f(\underset{\sim}{A},\underset{\sim}{B})(Z)=\bigwedge_{f(x,y)=z}(\underset{\sim}{A}(x)\vee\underset{\sim}{B}(y))$

Let **R** be real number field and $f:\mathbf{R}\to\mathbf{R}$ be a Riemann integrable function, then $\int_a^b f(x)dx$ can be seen as a function:

$I:\mathbf{R}\times\mathbf{R}\to\mathbf{R}$

$(a,b)\to I(a,b)\triangleq\int_a^b f(x)dx$

By the use of extension principle and lemma 2, we have:

$I:\mathscr{F}(\mathbf{R})\times\mathscr{F}(\mathbf{R})\to\mathscr{F}(\mathbf{R})$

$(\underset{\sim}{A},\underset{\sim}{B})\to I(\underset{\sim}{A},\underset{\sim}{B})\triangleq\int_{\underset{\sim}{A}}^{\underset{\sim}{B}}f(x)dx\ \bigcap_{\lambda\in[0,1]}\lambda\int_{A^{\lambda}}^{B^{\lambda}}f(x)dx$

where $\int_{A^{\lambda}}^{B^{\lambda}}f(x)dx\triangleq\{z|\exists\ (a,b)\in A^{\lambda}\times B^{\lambda}, z=\int_a^b f(x)dx\}$

then we have:

Theorem 1 $$(\int_{\underset{\sim}{A}}^{\underset{\sim}{B}}f(x)dx)(z)=\bigwedge_{\int_a^b f(x)dx=z}(\underset{\sim}{A}(a)\vee\underset{\sim}{B}(b)) \qquad (1)$$

Definition 1 Formula (I) is called fuzzy Riemann integral of function f over $(\underset{\sim}{A},\underset{\sim}{B})$.

3. Contave fuzzy numbers and fuzzy Riemann integral of contave fuzzy number.

Definition 2 Let $\underset{\sim}{A}\in \mathscr{F}(R)$, if for any $\lambda \in [0,1]$, we have:

(1) $A^\lambda = [a_-^\lambda, a_+^\lambda]$ (where $a_-^\lambda, a_+^\lambda \in \mathbf{R}$ and $a_-^\lambda \leqslant a_+^\lambda$)

(2) $A^0 \neq \varnothing$

then $\underset{\sim}{A}$ is called as a contave fuzzy number. $\underset{\sim}{R}$ is denoted as a set of all contave fuzzy numbers.

Theorem 2 $\underset{\sim}{A}$ be a contave fuzzy number if and only if

$$\underset{\sim}{A}(x) = \begin{cases} 0 & x \in [m,n] \neq \varnothing \\ L(x) & x < m \\ R(x) & x > n \end{cases}$$

where $L(x)$ is a decreasing and left continuous function, $0 \leqslant L(x) \leqslant 1$ and $\lim\limits_{x\to\infty} L(x) = 1$; $R(x)$ is a increasing and right continuous function, $0 \leqslant R(x) \leqslant 1$ and $\lim\limits_{x\to\infty} R(x) = 1$.

We denote $\underset{\sim}{A}$ as $\underset{\sim}{A} = ([m_A, n_A], L_A, R_A)$

Let $\underset{\sim}{R} = \{[a,b] \mid a,b \in \mathbf{R}, a \leqslant b\}$, then we have:

Theorem 3 Let $H: [0,1] \to \underset{\sim}{R}$ $\lambda \to H(\lambda) = [m^\lambda, n^\lambda] (\neq \varnothing)$ be a order set embedding over $[0,1]$, then

(1) $\underset{\sim}{A} = \bigcap\limits_{\lambda\in[0,1]} \lambda H(\lambda) \in \underset{\sim}{R}$

(2) $A^\lambda = \bigcap\limits_{n=1}^{\infty} H(\lambda_n) \qquad (\lambda_n = (1+\frac{1}{n})\lambda)$

(3) $\underset{\sim}{A} = ([m_A, n_A], L_A, R_A)$

where $\quad m_A = \lim\limits_{n\to\infty} m_{\lambda_n}, \quad n_A = \lim\limits_{n\to\infty} n_{\lambda_n} \quad (\lambda_n = 1 - \frac{1}{n})$

$$L_A(x) = \bigwedge_{\lambda\in[0,1]} \{m_\lambda \leqslant x\}, \quad R_A(x) = \bigwedge_{\lambda\in[0,1]} \{\lambda \mid n_\lambda \geqslant x\}$$

Theorem 4 Let $f: \mathbf{R}^n \to \mathbf{R}$ be a continuous function, $\underset{\sim}{A}_k \in \underset{\sim}{R}$, then $f(\underset{\sim}{A}_1, \underset{\sim}{A}_2 \cdots \underset{\sim}{A}_n)^\lambda = f(A_1^\lambda, A_2^\lambda, \cdots, A_n^\lambda)$

Corollary Let $\underset{\sim}{A}, \underset{\sim}{B} \in \mathbf{R}$, then for any $\lambda \in [0,1]$,

$(\underset{\sim}{A} + \underset{\sim}{B})^\lambda = A^\lambda \pm B^\lambda \qquad (A \cdot B)^\lambda = A^\lambda \cdot B^\lambda$

$(\underset{\sim}{A} \div \underset{\sim}{B})^\lambda = A^\lambda \div B^\lambda \qquad (k\underset{\sim}{A})^\lambda = kA^\lambda \quad (k \in \mathbf{R})$

Definition 3 (1) Function $\tilde{f}: [a,b] \to \underset{\sim}{R}$ is called as a fuzzy value function over $[a,b]$.

(2) $f^\lambda(x) \triangleq [\tilde{f}(x)]^\lambda$ is called as λ — lower cut function of $\tilde{f}$.

Definition 4 Let $\tilde{f}: [a,b] \to \underset{\sim}{R}$ be a fuzzy value function and $f^{\lambda}(x) = [f_{-}^{\lambda}(x), f_{+}^{\lambda}(x)]$ is integrable over $[a,b]$ (i.e., $f_{-}^{\lambda}(x), f_{+}^{\lambda}(x)$ is a Riemann integrable function). Integral of $\tilde{f}$ over $[a,b]$ is denoted as

$$\int_a^b \tilde{f}(x)dx \triangleq \bigcap_{\lambda \in [0,1]} \lambda \int_a^b f^{\lambda}(x)dx = \bigcap_{\lambda \in [0,1]} \lambda [\int_a^b f_{-}^{\lambda}(x)dx, \int_a^b f_{+}^{\lambda}(x)dx]$$

Theorem 5 If $\tilde{f}$ is a Riemann integrable over $[a,b]$, then $\int_a^b \tilde{f}(x)dx \in \underset{\sim}{R}$ and

$$(\int_a^b \tilde{f}(x)dx)^{\lambda} = \bigcap_{n=1}^{\infty} \int_a^b f^{\lambda_n}(x)dx \quad (\lambda_n = (1 + \frac{1}{n})\lambda)$$

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