Topology of interval-valued fuzzy sets

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Abstract: In this paper topology of interval-valued fuzzy sets is defined and some of its properties are studied. It is shown that the category of topological spaces of interval-valued fuzzy sets and continuous functions forms a topological category.

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0. Introduction

Let $D[0,1]$ be the set of all closed subintervals of the interval $[0,1]$. The elements of $D[0,1]$ are generally denoted by capital letters $M,N,\ldots\ldots$ It is known that $M = [M^L, M^U]$, where M^L and M^U are respectively the lower and the upper end points; and $M = N$ iff $M^L = N^L$ and $M^U = N^U$. Further $M \leq N$ iff $M^L \leq N^L$ and $M^U \leq N^U$. The complementary of M is denoted by M^c and is defined by $M^c = 1 - M = [1 - M^U, 1 - M^L]$.

Let X be a given nonempty set. Following Zadeh [6], a function $A : X \rightarrow D[o,1]$ is called an interval-valued fuzzy set (briefly it is written as IVF set) on X.

Thus for each $x \in X$, $A(x)$ is a closed interval whose lower and upper end points are denoted by $[A(x)]^L$ and $[A(x)]^U$ respectively. Clearly every fuzzy set (briefly denoted by FS) on X is an IVF set on X. For any interval $[a,b]$ ($\subset [0,1]$), the IVF set whose value is the interval [a,b] for all $x \in X$, is denoted by $\widetilde{[a,b]}$. In particular, if a=b the IVF set $\widetilde{[a,b]}$ is denoted by simply $\tilde{a}$.

For a particular $x_0 \in X$ and for a particular interval $[a,b] \in D[0,1]$ with b>0, the IVF set which takes the value [a,b] at x_0 and 0 elsewhere in X is called an interval-valued fuzzy point (briefly it is called IVF point) and is denoted by $[a,b]_{x_0}$. In particular if b=a, then it is also denoted by a_{x_0}.

In [2], C.L.Chang defined topology on fuzzy sets (he called fuzzy topology). Later many authors (see [4],[5]) have studied with this topology of fuzzy sets.
In this paper we shall define topology of interval-valued fuzzy sets and study some topological properties.

In section 1 we give some preliminary results on IVF sets.
In section 2 we give the definition of a topology on IVF sets on X, and obtain some basic results. We define product topology and establish that the class of all topological spaces of IVF sets and continuous functions forms a topological category. We also define compactness and obtain the Alexander theorem in IVF setting.
In section 3 we define topology on an IVF set and continuous function from an IVF set to an IVF set. We prove the continuous image of compact IVF set is compact.

When no confusion is likely to arise we shall use D in place of $D[0,1]$ (defined earlier). Then D^X denotes the set of all IVF sets on X. Thus for $M \in D$, M_x is an IVF point on X.

1. Some preliminaries

Let $A, B \in D^X$. The equality of two IVF sets is defined by
$A = B \Leftrightarrow [A(x)]^L = [B(x)]^L$ and $[A(x)]^U = [B(x)]^U, \forall x \in X$.

Subset relation is defined by
$A \subset B \Leftrightarrow [A(x)]^L \leq [B(x)]^L$ and $[A(x)]^U \leq [B(x)]^U, \forall x \in X$.

The complement A^c of A is defined by
$[A^c(x)]^L = 1 - [A(x)]^U$ and $[A^c(x)]^U = 1 - [A(x)]^L, \forall x \in X$.

For a family of IVF sets $\{A_i; i \in I\}$, the union $G = \bigcup_i A_i$ and the intersection $F = \bigcap_i A_i$ are respectively defined by

$[G(x)]^L = Max_i[A_i(x)]^L$, $[G(x)]^U = Max_i[A_i(x)]^U, \forall\, x \in X$

and $[F(x)]^L = Min_i[A_i(x)]^L$, $[F(x)]^U = Min_i[A_i(x)]^U, \forall\, x \in X$.

For $A \in D^X$, the set $\{x \in X; A^U(x) > 0\}$ is called the support of A and is denoted by A_+.

An IVF point M_x is said to belong to an IVF set A (this is symbolically denoted by $M_x \tilde{\in} A$) if $[A(x)]^L \geq M^L$ and $[A(x)]^U \geq M^U$. It can be easily shown that $A = \bigcup\{M_x;\ M_x \tilde{\in} A\}$.

Theorem 1.1. *For all* $A, B, C, A_i, B_i \in D^X$ *followings hold:*
(i) $\tilde{0} \subset A \subset \tilde{1}$,
(ii) $A \cup B = B \cup A$; $A \cap B = B \cap A$,
(iii) $A \cup (B \cup C) = (A \cup B) \cup C$; $A \cap (B \cap C) = (A \cap B) \cap C$,
(iv) $A, B \subset A \cup B$; $A \cap B \subset A, B$,
(v) $A \cap (\bigcup_i B_i) = \bigcup_i (A \cap B_i)$,
(vi) $A \cup (\bigcap_i B_i) = \bigcap_i (A \cup B_i)$,
(vii) $(\tilde{0})^c = \tilde{1}$; $(\tilde{1})^c = \tilde{0}$,
(viii) $(A^c)^c = A$,
(ix) $(\bigcup_i A_i)^c = \bigcap_i A_i^c$,
(x) $(\bigcap_i A_i)^c = \bigcup_i A_i^c$.

Remark 1.2. In ordinary fuzzy setting, a fuzzy point $P_x \tilde{\in} \lambda_1 \cup \lambda_2$ iff $P_x \tilde{\in} \lambda_1$ or $P_x \tilde{\in} \lambda_2$ or both. But this is not true in IVF setting. This is shown by the following example:

Example 1.3. Let $X = \{x_1, x_2\}$,
$A_1 = \widetilde{[\frac{1}{4}, \frac{1}{2}]}$, $A_2 = \widetilde{[0, \frac{3}{4}]}$. Then the IVF point $M_{x_1} (= [\frac{1}{4}, \frac{2}{3}]_{x_1}) \tilde{\in} A_1 \cup A_2$ but $M_{x_1} \tilde{\notin} A_1$ and $M_{x_1} \tilde{\notin} A_2$.

Definition 1.4. Let $f : X \to Y$ be a function. Let $\lambda \in D^X$. Then the image of λ written as $f(\lambda)$ is defined by

$$[f(\lambda)(y)]^L = \begin{cases} Sup_{y=f(x)}\{[\lambda(x)]^L\}, & \text{if } f^{-1}(y) \text{ is not empty} \\ 0, & \text{otherwise} \end{cases}$$

$$[f(\lambda)(y)]^U = \begin{cases} Sup_{y=f(x)}\{[\lambda(x)]^U\}, & \text{if } f^{-1}(y) \text{ is not empty} \\ 0, & \text{otherwise} \end{cases}$$

Let $\mu \in D^Y$. Then the inverse of μ written as $f^{-1}(\mu)$ is defined by

$[f^{-1}(\mu)(x)]^L = [\mu(f(x))]^L$ and $[f^{-1}(\mu)(x)]^U = [\mu(f(x))]^U, \forall x \in X$.

Theorem 1.5. *Let $f : X \to Y$ be a function. Then*

(i) $f^{-1}(B^c) = [f^{-1}(B)]^c, \forall B \in D^Y$,

(ii) $[f(A)]^c \subset f(A^c), \forall A \in D^X$,

(iii) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$, *where* $B_1, B_2 \in D^Y$,

(iv) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$, *where* $A_1, A_2 \in D^X$,

(v) $f(f^{-1}(B)) \subset B, \forall B \in D^Y$,

(vi) $A \subset f^{-1}(f(A)), \forall A \in D^X$,

(vii) *Let f be a function from X to Y and g be a function from Y to Z. Then* $(gof)^{-1}(C) = f^{-1}(g^{-1}(C)), \forall C \in D^Z$, *where gof is the composition of g and f,*

(viii) $f^{-1}(\bigcup_{i\in\Delta} B_i) = \bigcup_{i\in\Delta} f^{-1}(B_i)$, $B_i \in D^Y$,

(ix) $f^{-1}(\bigcap_{i\in\Delta} B_i) = \bigcap_{i\in\Delta} f^{-1}(B_i)$, $B_i \in D^Y$.

2. Topology of interval-valued fuzzy sets

In this section we shall first give the definition of topology of interval- valued fuzzy sets.

Definition 2.1. A topological space of IVF sets is a pair (X, τ) where X is a nonempty set and τ is a family of IVF sets on X satisfying the following three axioms:

(1) $\tilde{1}, \tilde{0} \in \tau$,

(2) $A, B \in \tau \Rightarrow A \bigcap B \in \tau$,

(3) $A_i \in \tau, i \in \Delta \Rightarrow \bigcup_{i\in\Delta} A_i \in \tau$.

τ is called a topology of IVF sets on X. Every member of τ is called IVF open. $B \in D^X$ is said to be closed in (X, τ) iff $B^c \in \tau$. As in ordinary topologies the indiscrete topology of IVF sets contains only $\tilde{1}$ and $\tilde{0}$, while the discrete topology of IVF sets contains all IVF sets. These two topologies are respectively denoted by τ^o and τ^1. A topology τ_1 is said to be weaker (or coarser) than a topology τ_2 iff $\tau_1 \subset \tau_2$. In that case τ_2 is said to be stronger (or finer) than τ_1.

Theorem 2.2. *Let $\{\tau_i : i \in I\}$ be a family of topologies of IVF sets on X. Then $\bigcap_i\{\tau_i : i \in I\}$ is also a topology of IVF sets on X.*

Theorem 2.3. *Let $\{\tau_i : i \in I\}$ be a family of topologies of IVF sets on X. Then $\{\tau_i : i \in I\}$ forms a complete lattice with respect to set inclusion relation of which τ^0 is the smallest element and τ^1 is the largest element.*

Theorem 2.4. *Let (X, τ) be a topological space of IVF sets and let T denote the collection of all IVF closed sets. Then*
(1) $\tilde{0}, \tilde{1} \in T$,
(2) $F_1, F_2 \in T \Rightarrow F_1 \bigcup F_2 \in T$,
(3) $F_i \in T, i \in \Delta \Rightarrow \bigcap_{i \in \Delta} F_i \in T$.

Definition 2.5. Let (X, τ) be a topological space of IVF sets. A subcollection $\mathcal{B}$ of τ is said to be a base for τ if every member of τ can be expressed as a union of members of $\mathcal{B}$.

Definition 2.6. Let (X, τ) be a topological space of IVF sets. A subcollection $\mathcal{S}$ of τ is said to be a subbase for τ if the family of all finite intersections of members of $\mathcal{S}$ forms a base for τ.

Theorem 2.7. *Let $\mathcal{B}$ be a family of IVF sets on X such that $\tilde{0}, \tilde{1} \in \mathcal{B}$. If for any $B_1, B_2 \in \mathcal{B}$ and for any $M_x \tilde{\in} B_1 \bigcap B_2$ there exists $w \in \mathcal{B}$ such that $M_x \tilde{\in} w \subset B_1 \bigcap B_2$, then $\mathcal{B}$ is a base for some topology of IVF sets on X.*

Remark 2.8. The converse of the above theorem is however not true. This is shown by the following example:

Example 2.9. Let X be a nonempty set.
Let $\tau = \{\tilde{0}, \tilde{1}, A_1, A_2, B_1, B_2, A_1 \bigcup A_2, A_1 \bigcap A_2, B_1 \bigcup B_2\}$, where $A_1 = \widetilde{[\frac{1}{4}, \frac{1}{2}]}$, $A_2 = \widetilde{[0, \frac{3}{4}]}$, $A_1 \bigcup A_2 = \widetilde{[\frac{1}{4}, \frac{3}{4}]}$, $A_1 \bigcap A_2 = \widetilde{[0, \frac{1}{2}]}$, $B_1 = \widetilde{[\frac{1}{4}, \frac{4}{5}]}$, $B_2 = \widetilde{[\frac{1}{2}, \frac{3}{4}]}$, $B_1 \bigcup B_2 = \widetilde{[\frac{1}{2}, \frac{4}{5}]}$, $B_1 \bigcap B_2 = \widetilde{[\frac{1}{4}, \frac{3}{4}]} = A_1 \bigcup A_2$. Then τ is a topology of IVF sets on X. Let $\mathcal{B} = \{\tilde{0}, \tilde{1}, A_1, A_2, B_1, B_2, A_1 \bigcap A_2\}$. Clearly $\mathcal{B}$ is a base for the topology τ. For the IVF point $[\frac{1}{4}, \frac{2}{3}]_x$ in X, $[\frac{1}{4}, \frac{2}{3}]_x \tilde{\in} B_1 \bigcap B_2$, but there exists no member w of $\mathcal{B}$ such that $[\frac{1}{4}, \frac{2}{3}]_x \tilde{\in} w \subset B_1 \bigcap B_2$.

Definition 2.10. An IVF set B in a topological space of IVF sets (X,τ) is said to be a neighbourhood (in short nbd) of an IVF point M_x iff there exists an IVF open set O such that $M_x \tilde{\in} O \subset B$.

Theorem 2.11. *In a topological space of IVF sets (X,τ) an IVF set $A \in \tau$ iff it is a nbd of each of its IVF points.*

Definition 2.12. Let $0 \le a \le b$ and $(a,b) \neq (0,0)$. A pair of numbers (δ,η) with the property
(i) $\delta \ge 0, \eta > 0$,
(ii) $\delta = 0$ iff $a = 0$,
(iii) $0 \le a - \delta < b - \eta$, is called an admissible pair.

Theorem 2.13. *Let (X,τ) be a topological space of IVF sets. For each IVF point M_x, let $N(M_x)$ denotes the collection of all nbds of M_x. Then*
(N1) $\tilde{1} \in N(M_x)$, $\forall\ M_x$ *and* $A \in N(M_x) \Rightarrow M_x \tilde{\in} A$,
(N2) $A, B \in N(M_x) \Rightarrow A \cap B \in N(M_x)$,
(N3) $A \subset B$ *and* $A \in N(M_x) \Rightarrow B \in N(M_x)$,
(N4) $A \in N([a-\delta, b-\eta]_x)$, *for all admissible* $\delta, \eta \Rightarrow A \in N([a,b]_x)$,
(N5) $A \in N(M_\xi)$, $B \in N(P_\xi) \Rightarrow A \cup B \in N(M_\xi \cup P_\xi)$,
(N6) $A \in N(M_x) \Rightarrow \exists\ S \in N(M_x)$ *such that* $S \subset A$ *and* $S \in N(P_\xi)$, $\forall\ P_\xi \tilde{\in} S$.

Theorem 2.14. *Let X be a nonempty set. Let for each IVF point M_x there exists a nonempty collection $N(M_x)$ of IVF sets on X satisfying $(N1)-(N6)$.*
Let $\tau = \{A \in D^X;\ A \in N(P_\xi),\ \forall\ P_\xi \tilde{\in} A\}$.
Then τ is a topology of IVF sets on X such that $N(M_x)$ is the family of all nbds of M_x in (X,τ).

Definition 2.15. Let A be an IVF set in a topological space of IVF sets (X,τ). Then an IVF point M_x is said to be an interior point of A iff A is a nbd of M_x.
The union of all interior points of A is called the interior of A and is denoted by int$A$.

Theorem 2.16. *Let A be an IVF set in a topological space of IVF sets (X,τ). Then int$A$ is open and is the largest IVF open set contained in A. An IVF set A is open iff A=int$A$.*

Definition 2.17. Let A be an IVF set in a topological space of IVF sets (X,τ). Define closure of A by $\bigcap\{B \in D^X;\ B$ is closed and $A \subset B\}$ and denote it by cl$A$.

Theorem 2.18.

(i) $cl\phi = \phi$,

(ii) $clX = X$,

(iii) $A \subset clA, \forall A \in D^X$,

(iv) clA *is a closed set,*

(v) A *is closed iff* $A = clA$,

(vi) $cl(A \bigcup B) = clA \bigcup clB$,

(vii) $cl(clA) = clA$.

Theorem 2.19. *For any* $A \in D^X$, $clA = [int(A^c)]^c$.

Definition 2.20. A function $f : (X,\tau_1) \to (Y,\tau_2)$ is said to be continuous if $f^{-1}(\mu) \in \tau_1$, $\forall\ \mu \in \tau_2$. If $f : (X,\tau_1) \to (Y,\tau_2)$ and $g : (Y,\tau_2) \to (Z,\tau_3)$ are continuous then gof is also continuous because of $(gof)^{-1}(C) = f^{-1}[g^{-1}(C)]$, $\forall\ C \in \tau_3$.

Theorem 2.21. *If* (X,τ_1) *and* (Y,τ_2) *be two topological spaces of IVF sets and* f *be a function from* X *to* Y, *then the following statements are equivalent:*

(a) the function f *is continuous,*

(b) the inverse of every IVF closed set is closed,

(c) for each IVF point M_x *in* X *the inverse of every nbd of* $f(M_x)$ *under* f *is a nbd of* M_x,

(d) for each IVF point M_x *in* X *and each nbd* V *of* $f(M_x)$, *there is a nbd* W *of* M_x *such that* $f(W) \subset V$,

(e) $f(clA) \subset cl(f(A))$.

Theorem 2.22. *Let* S *be a family of IVF sets of* X *such that* $\tilde{0}, \tilde{1} \in S$. *Then* S *is a subbase for the topology* τ, *whose members are of the form* $\bigcup_{i\in\Delta}(\bigcap_{k\in J_i} S_{i,k})$ *where* Δ *is an arbitrary index set and for each* $i \in \Delta$, J_i *is a finite index set,* $S_{i,k} \in S$ *for* $i \in \Delta$ *and* $k \in J_i$.

Definition 2.23. Let $\{f_i : X \to (Y_i,\tau_i)\}_{i\in\Delta}$ be a family of functions where X is a nonempty set and $\{(Y_i,\tau_i)\}_{i\in\Delta}$ is a family of topological spaces of IVF sets. Then the topology τ generated from the subbase $S = \{f_i^{-1}(O);\ O \in \tau_i, i \in \Delta\}$ is called the topology of IVF sets (or initial topology of IVF sets on X) induced by the family of functions $\{f_i\}_{i\in\Delta}$ from the family of topological spaces of IVF sets $\{(Y_i, \tau_i)\}_{i\in\Delta}$.

Theorem 2.24. *The initial topology of IVF sets on X induced by the family* $\{f_i : X \to (Y_i, \tau_i)\}_{i\in\Delta}$ *is the coarsest topology of IVF sets on X with respect to which each* $f_i : (X, \tau) \to (Y_i, \tau_i)$ *is continuous,* $i \in \Delta$.

Definition 2.25. Let $\{(X_i, \tau_i)\}_{i\in\Delta}$ be a family of topological spaces of IVF sets. Then the initial topology of IVF sets on X $(= \Pi_{i\in\Delta} X_i)$ generated by the family $\{p_i : X \to (X_i, \tau_i)\}_{i\in\Delta}$ is called the product topology on X.
(here $p_i : X \to X_i$ is the projection mapping, $i \in \Delta$.)
This topology may be denoted by $\Pi_{i\in\Delta}\tau_i$.

Remark 2.26. Clearly, by definition of the product topology of IVF sets τ on the product space $X(= \Pi_{\alpha\in\Delta} X_\alpha)$, the projection mappings $P_\alpha : (X, \tau) \to (X_\alpha, \tau_\alpha)$ are continuous.
However, the projection mappings are not necessarily open. This is shown by the following example:

Example 2.27. Let $X_1 = \{x_1, x_2\} = X_2$ and $\tau_1 = \{\tilde{O}_{X_1}, \tilde{1}_{X_1}, \lambda\}$, $\tau_2 = \{\tilde{O}_{X_2}, \tilde{1}_{X_2}, \mu\}$, where, $\lambda(x_1) = 0.4, \lambda(x_2) = 0.6; \mu(x_1) = 0.8,\ \mu(x_2) = 0.3$. Let $\tau = \{\tilde{O}_{X_1\times X_2}, \tilde{1}_{X_1\times X_2}, u_1, u_2, u_1 \cap u_2, u_1 \cup u_2\}$, where, $u_1 = p_1^{-1}(\lambda), u_2 = p_2^{-1}(\mu)$.
Then τ is the product topology on $X_1 \times X_2$ and $p_2(u_1) \notin \tau_2$, $p_1(u_2) \notin \tau_1$. Thus the projection mappings p_1 and p_2 are not open.

Theorem 2.28. *Let* $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}$ *be a family of topological spaces of IVF sets and* τ *be the product topology on* $X(= \Pi_{\alpha\in\Delta} X_\alpha)$. *Let* (Y, τ') *be a topological space of IVF sets and* $f : Y \to X$ *be a mapping. Then* $f : (Y, \tau') \to (X, \tau)$ *is continuous iff* $p_\alpha o f : (Y, \tau') \to (X_\alpha, \tau_\alpha)$ *is continuous,* $\forall\ \alpha \in \Delta$.

Remark 2.29. From theorem 2.28, it thus follows that the class of all topological spaces of IVF sets and continuous function forms a topological category.

Definition 2.30. Let (X, τ) be a topological space of IVF sets. Then τ is said to be Lowen-type if every constant IVF set on X belongs to τ.

Theorem 2.31. *Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}$ be a family of Lowen-type topological spaces of IVF sets and $X = \Pi_{\alpha\in\Delta}X_\alpha$, $\tau = \Pi_{\alpha\in\Delta}\tau_\alpha$. Then the projection mappings $p_\alpha : (X, \tau) \to (X_\alpha, \tau_\alpha)$, $\alpha \in \Delta$ are open mappings.*

Definition 2.32. A family C of IVF sets is said to be a cover of X if $\tilde{1}_X \subset \bigcup_{w\in C} w$. It is called an open cover of X iff C is a cover of X and each member of C is an IVF open set. A subcover of C is a subfamily of C which is also a cover.

Definition 2.33. A topological space of IVF sets (X, τ) is said to be compact iff each open cover of X has a finite subcover.

We shall now establish the Alexander theorem on compactness in IVF setting.

Theorem 2.34. *If S is a subbase for a topology of IVF sets on X such that every cover of X by members of S has a finite subcover, then X is compact.*

Theorem 2.35. *Let $\{(X_i, \tau_i)\}_{i=1}^n$ be a finite collection of IVF spaces, $X = \Pi_{i=1}^n X_i$, $\tau = \Pi_{i=1}^n \tau_i$. Then (X, τ) is compact.*

However, the above theorem is not true when the number of spaces is infinite. This can be shown by the following example:

Example 2.36. Let $X_i = [0,1]$, $i \in N(N = 1, 2, 3, ...)$, $\tau_i = \{\tilde{0}, \tilde{1}, \lambda_i\}$, where $\lambda_i(x) = \frac{i}{i+1}, \forall\, x \in [0,1]$. Then (X_i, τ_i) are compact spaces of fuzzy sets.
Let, $X = \Pi_{i=1}^\infty X_i$, $\tau = \Pi_{i=1}^\infty \tau_i$. Then, $[\vee_{i=1}^\infty(p_i^{-1}(\lambda_i))](x) = \vee_{i=1}^\infty(\lambda_i p_i(x)) = \vee_{i=1}^\infty \frac{i}{i+1} = 1$, $\forall\, x \in X$. So, $\{p_i^{-1}(\lambda_i)\}_{i=1}^\infty$ is an open covering of X; but $\{p_i^{-1}(\lambda_i)\}_{i=1}^\infty$ has no finite subcovering, since for $\xi = (\frac{1}{2}, \frac{1}{2}, ...) \in X$ and for finite subset J of N, $\vee_{i\in J} p_i^{-1}(\lambda_i)(\xi) < 1$. Hence (X, τ) is not compact.

3. Topology on an interval-valued fuzzy set

In [1], M.K.Chakraborty and T.M.G.Ahsanullah gave the definition of a topology (they called it fuzzy topology) on a fuzzy set. In this section we give the definition of a topology on an interval-valued fuzzy set.

Let X and Y be two nonempty crisp sets. Let $A \in D^X, B \in D^Y$, $\lambda \subset A, \mu \subset B$. The supports of A and B are denoted by A_+ and B_+ respectively.
With these notations we now give the following definition:

Definition 3.1. Let $f : A_+ \to B_+$ be a mapping. If $Bf(x) \geq A(x)$, $\forall\, x \in A_+$, then we say f is a mapping from A to B. We also denote it by $f : A \to B$. For $f : A \to B$ and for $\lambda \subset A$, we define $f(\lambda)$ by

$$[f(\lambda)(y)]^L = \begin{cases} \vee_{f(x)=y}[\lambda(x)]^L, & \text{if } f^{-1}(y) \text{ is not empty} \\ 0, & \text{if } f^{-1}(y) \text{ is empty} \end{cases} \qquad y \in B_+.$$

and

$$[f(\lambda)(y)]^U = \begin{cases} \vee_{f(x)=y}[\lambda(x)]^U, & \text{if } f^{-1}(y) \text{ is not empty} \\ 0, & \text{if } f^{-1}(y) \text{ is empty} \end{cases} \qquad y \in B_+.$$

For $\mu \subset B$, we define $f^{-1}(\mu)$ by
$[f^{-1}(\mu)(x)]^L = [A(x)]^L \wedge [\mu(f(x))]^L$, $x \in A_+$
and
$[f^{-1}(\mu)(x)]^U = [A(x)]^U \wedge [\mu(f(x))]^U$, $x \in A_+$.
With the above notations following holds:

Theorem 3.2. *(i)* $f(\lambda) \subset B, f^{-1}(\mu) \subset A$,
(ii) $f(\vee_i \lambda_i) = \vee_i f(\lambda_i), \lambda_i \subset A, i \in \Delta$,
(iii) $\lambda_1 \subset \lambda_2 \Rightarrow f(\lambda_1) \subset f(\lambda_2)$, $\lambda_1, \lambda_2 \subset A$,
(iv) $f^{-1}(\vee_i \mu_i) = \vee_i f^{-1}(\mu_i), \mu_i \subset B, i \in \Delta$,
(v) $f(f^{-1}(\mu)) \subset \mu, \mu \subset B$,
(vi) $\lambda \subset f^{-1}f(\lambda), \lambda \subset A$.

Definition 3.3. Let $X(\neq \phi)$ be a crisp set and $A \in D^X$. A subset T of D^X is said to be a topology on A if
(1) $\lambda \in T \Rightarrow \lambda \subset A$,
(2) $\tilde{0}, A \in T$,

(3) $\lambda_1, \lambda_2 \in T \Rightarrow \lambda_1 \bigcap \lambda_2 \in T$,
(4) $\lambda_i \in T, i \in \Delta \Rightarrow \bigcup_{i \in \Delta} \lambda_i \in T$.

If τ is a topology of IVF sets on X and if $\lambda \in D^X$, then, $\tau_\lambda = \{\lambda \bigcap \mu;\ \mu \in \tau\}$ is a topology on λ.

Let X and Y be two nonempty crisp sets and $\phi : X \rightarrow Y$ be a mapping, $A \in D^X$ and $B = \phi(A)$. Let ϕ_A denote the restriction of ϕ on A_+. Then ϕ_A is a function from A to B, since $B\phi_A(x) \geq A(x), \forall\, x \in A_+$.

Theorem 3.4. *Let (X, τ) and (Y, τ') be two IVF topological spaces and $\phi : (X, \tau) \rightarrow (Y, \tau')$ be continuous. Let $A \in D^X$ and $B = \phi(A)$. Then $\phi_A : (A, \tau_A) \rightarrow (B, \tau'_B)$ is continuous.*

Definition 3.5. Let X and Y be two crisp sets, $A \in D^X, B \in D^Y$. Let τ and τ' be topologies on A and B respectively and $f : A \rightarrow B$ be a mapping. Then f is said to be continuous iff $f^{-1}(\mu) \in \tau,\ \forall\, \mu \in \tau'$.

Definition 3.6. Let λ be an IVF set on X. Then a family $\mathcal{A}$ of IVF sets on X is said to be a cover of λ if $\lambda \subseteq \bigcup_{w \in \mathcal{A}} w$.

Definition 3.7. Let $X(\neq \phi)$ be a crisp set. Let $\lambda \in D^X$ and T be a topology on λ. Let $\mu \subset \lambda$. Then μ is said to be compact if every open cover of μ by members of T has a finite subcover.

Theorem 3.8. *Let $A \in D^X, B \in D^Y$, where X and Y are nonempty sets. Let τ and τ' be topologies on A and B respectively and f be a continuous mapping from (A, τ) to (B, τ'). Let λ $(\subset A)$ be compact. Then $f(\lambda)$ is compact.*

References

[1] M.K.Chakraborty and T.M.G. Ahsanullah, Fuzzy topology on fuzzy sets and tolerance topology, Fuzzy Sets and Systems 45(1992) 103-108.

[2] C.L.Chang, Fuzzy topological spaces, J.Math.Anal.Appl. 24(1968) 182-190.

[3] J.L.Kelley, General Topology, Van Nostrand Reinhold Company, pp 139.

[4] R.Lowen, Fuzzy topological spaces and fuzzy compactness, J.Math.Anal.Appl. 56(1976), 621-633. MR55 # 13357; Zbl.342 # 54003.

[5] Pu Pao-Ming and Liu Ying-Ming, Fuzzy topology I, Neighbourhood structure of a fuzzy point and Moore-Smith convergence, J.Math.Anal.Appl. 76(1980) 571-599.

[6] L.A.Zadeh, The concept of a linguistic variable and its application to approximate reasoning-I.Inform.Sci. 8(1975) 199-249.