ON HAUSDORFF FRAME-FUZZY TOPOLOGICAL SPACES

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ABSTRACT

In the paper, a valid definition of pointless T_2 in Frame-fuzzy topological space is introduced, and the relationship between several kinds of Frame-fuzzy topological space T_2 separation axioms and the conditions of their application are discussed.

Keywords: Prime, Hausdorff, Spatial frame, Frame-fuzzy topological space

Throughout this paper L is a frame. We will denote by $pr(L)$ the set of primes of L, by (L^X,δ) the Frame-fuzzy topological space. $\forall x \in X, \lambda \in pr(L)$, A_λ^x: $X \to L$ is the L-fuzzy set defined by $A_\lambda^x(x)=\lambda$ and $A_\lambda^x(y)=1$ if $y \neq x$. $\neg x = \vee\{b \in L: b \wedge x = 0\}$. We say $a \lessdot b$ if there exists $c \in L$ with $c \wedge a=0$ and $c \vee b=1$; L is spatial iff for all $a, b \in L$, $a \not\leq b$, there exists $\lambda \in pr(L)$ with $a \not\leq \lambda$ and $b \leq \lambda$.[1]

1. Pointless Hausdorffness in Frame-fuzzy topological space

Definition 1.1 (L^X,δ) is T_2^* iff $A_\lambda^x = \vee\{B \in \delta: B \lessdot A_\lambda^x\}$ for each $x \in X$ and $\lambda \in pr(L)$.[2]

By this definition, however, all fuzzy topological spaces(i.e. $L=I$) aren't T_2^*. In fact, if (L^X,δ) is T_2^*, then L must satisfy some separation axioms.

Definition 1.2 (1) L is T_2^* iff $\lambda = \vee\{p \in L: p \lessdot \lambda\}$ for each $\lambda \in pr(L)$.

(2) L is T_2^{**} iff for every $r_1, r_2 \in pr(L)$ with $r_1 \neq r_2$, there exist $a, b \in L$ such that $a \not\leq r_1$, $b \not\leq r_2$ and $a \wedge b=0$.[3]

Theorem 1.3 For any non-empty set X, if $pr(L) \neq \varnothing$, then the following conditions are equivalent :

(1) There exists $\delta \subseteq L^X$ such that (L^X,δ) is T_2^* ;

(2) L is T_2^* ;

Morever, if L is spatial, then (1) and (2) are equivalent to the following condition:

(3) L is T_2^{**} .

Proof : (1)$\Leftrightarrow$(2) : obviously .

(2)$\Rightarrow$(3) : Assume that L isn't T_2^{**}. Then there exist η , $\lambda \in pr(L)$ with $\eta \neq \lambda$ (without loss of generality , let $\eta \not\leq \lambda$) , for any $a, b \in L$, $a \not\leq \eta$ and $b \not\leq \lambda$ imply $a \wedge b \neq 0$. If there exists $p \in L$ with $p \prec \eta$ and $p \not\leq \lambda$, then for each $q \in L$ with $q \wedge p=0$, we have $q \leq \eta$. So $\neg p \vee \eta \leq \eta \neq 1$. This contradicts with $p \prec \eta$. Thus $p \prec \eta$ implies $p \leq \lambda$. Hence $\vee\{p \in L: p \prec \eta\} \leq \lambda$ and $\vee\{p \in L: p \prec \eta\} \neq \eta$. This is a contradiction .

(3)$\Rightarrow$(2) : Let $\lambda' = \vee\{p \in L: p \prec \lambda\}$. Assume $\lambda' \neq \lambda$. Then there exists $\eta \in pr(L)$ such that $\lambda \not\leq \eta$, $\lambda' \leq \eta$. By T_2^{**}, there exist $a, b \in L$ such that $a \not\leq \lambda$, $b \not\leq \eta$ and $a \wedge b=0$. Thus $b \not\leq \lambda'$. On the other hand, if $a \vee \lambda = \lambda'' \neq 1$, then there exists $\lambda''' \in pr(L)$ such that $1 \not\leq \lambda'''$, $\lambda'' \leq \lambda'''$. This implies $\lambda \leq \lambda'''$, a contradiction . Hence $b \prec \lambda$ and $b \leq \lambda'$. This contradicts with $b \not\leq \lambda'$.

We have known that there exists a T_2^* topology on X iff L is T_2^*. However , when L is a pointless frame , even if δ is the trival topology on X, (L^X,δ) is T_2^* . Therefore the definition of T_2^* isn't satisfactory sometimes . We give the following definitions .

Definition 1.4 Let $a, b \in L$, $a \neq 1$. We say $b \lessapprox a$ iff $b \leq a$ and $\neg b \not\leq a$.

Definition 1.5 (L^X,δ) is a T_2^0 space if $A = \vee\{B \in \delta: B \lessapprox A\}$ for every $A \in \delta$.

Theorem 1.6 If L is spatial , and $A_\lambda^x \in \delta$ for each $\lambda \in pr(L)$ and $x \in X$ (i.e. (L^X,δ) is a T_1 space[4]), then the following conditions are equivalent :

(1) (L^X,δ) is a T_2^* space;

(2) (L^X,δ) is a T_2^0 space.

Proof : (1)$\Rightarrow$(2) Assume that (L^X,δ) is T_2^*, then $A_\lambda^x = \vee\{B \in \delta: B \prec A_\lambda^x\}$. Let $B \prec A_\lambda^x$, then there exists $C \in \delta$ such that $C \wedge B=0$, $C \vee A_\lambda^x=1$. Hence $\neg_\delta B \not\leq A_\lambda^x$. Thus $B \lessapprox A_\lambda^x$. If $A \neq 1_X$ for each $A \in \delta$, then there exists $x \in X$

such that $A(x)\neq 1$. Therefore there exists $\lambda\in pr(L)$ such that $A(x)\leq\lambda<1$. Thus $A\leq A_\lambda^x$. Since $B\precsim A_\lambda^x$, $A\wedge B\precsim A$. Hence $A=\vee\{A\wedge B\in\delta : A\wedge B\precsim A\}$. So (L^X,δ) is T_2^0 .

(2)$\Rightarrow$(1) For each $B\in\delta$ with $B\precsim A_\lambda^x$, let $B(x)=a\in L$ and denote $\neg_\delta B(x)=\neg a$. If $\lambda\vee\neg a=b\neq 1$, then $A_b^x=A_\lambda^x\vee B$. Thus $A_b^x\in\delta$ and $A_b^x=\vee\{D\in\delta: D\precsim A_b^x\}$. On the other hand , if $D(x)\not\leq\lambda$ for each $D\in\delta$ with $D\precsim A_b^x$, then by $D(x)\wedge\neg_\delta D(x)=0<\lambda$ and $\lambda\in pr(L)$, we have $\neg_\delta D(x)\leq\lambda$. Thus $\neg_\delta D\leq A_\lambda^x\leq A_b^x$. This contradicts with $D\precsim A_b^x$. Hence $D\precsim A_b^x$ implies $D(x)\leq\lambda$. So $\vee\{D\in\delta: D\precsim A_b^x\}\leq A_\lambda^x\neq A_b^x$, a contradiction. Hence $\lambda\vee\neg a=1$. Furthermore , take $C\in L^X$ satisfying $C(x)=\neg a$ and $C(y)=0$ for each $y(\neq x)$, then $C\leq\neg_\delta B$. From $(A_\lambda^x\vee C)(x)=\lambda\vee\neg a=1$, it follows $B\lessdot A_\lambda^x$. Hence (L^X,δ) is a T_2^* space .

2 Hausdorffness in L-fuzzy topological space and Frame-fuzzy topological space

In usual documents ([5]) , (L^X,δ) is a T_2 space if for any pair of distinct L-fuzzy points x_t and y_r $(x\neq y)$, there exist $U,V\in\delta$ such that $x_t\in U, y_r\in V$ and $U\cap V=0$.

Theorem 2.1 If $A_0^x=\vee\{B\in\delta : B\lessdot A_\lambda^x , B(x)=0\}$ for each $x\in X$ and $\lambda\in pr(L)$------(*), then (L^X,δ) is a T_2 space .

The proof is directed .

It is easy to see that the condition (*) of Theorem 2.1 is strictly weaker than T_2^* separation axiom .

Example 2.2 Let $X=\{a, b\}\cup N\times N$, where $a,b\notin N\times N$. Denote $L_i=N\times\{i\}$. Define $O_i(i=1,2,3)$ as follows :

$$O_1=\{A\in L^X: A(a)=A(b)=0\}\ ;$$

$$O_2=\{A\in L^X:(A(a)\neq 0)\wedge(\exists k\in N,\forall i\geq k,|\{x\in L_i: A(x)=0\}|<\aleph_0)\}$$

$$O_3=\{A\in L^X:(\exists n, A(b)\geq 1-\frac{1}{n})\wedge(\forall i\leq n,|\{x\in L_i: A(x)=0\}|<\aleph_0)\}.$$

Let τ be the topology generated by $O_1\cup O_2\cup O_3$.Then (I^X,τ) forms a fuzzy space . It is obvious that (I^X,τ) is a T_2 space . But for each $\lambda\in pr(L)$ and $\lambda>0$, we have $A_0^b\neq\vee\{A\in\tau : A\lessdot A_\lambda^b\}$. In fact , if $A(a)\neq 0$, then there exists $i\in N$ with $i=\wedge\{j\in N: A(x)\neq 0, x\in B\subset L_j, |B|\geq\aleph_0\}$. If $B\wedge A=0$

for every $B \in \tau$, then $B(b) < 1-\frac{1}{i}$. Since $A_\lambda^b(b) \neq 1$, $(A_\lambda^b \vee B)(b) \neq 1$. Hence $A \prec A_\lambda^b$ doesn't hold . This implies that if $A \prec A_\lambda^b$, then $A(a)=0$. Thus $\vee\{A \in \tau : A \prec A_\lambda^b\}(a)=0$. But $A_0^b(a)=1$, a contradiction.

By making use of the relation "$\tilde{<}$" introduced in the paper , a characterization of T_2 separation in (L^X, δ) is given . It also shows clearly the differences between T_2^* and T_2 in (L^X, δ) .

Theorem 2. 3 (L^X, δ) is T_2 iff $A_0^x = \vee\{B \in \delta : B \tilde{<} A_\lambda^x , B(x)=0\}$ for each $x \in X$ and $\lambda \in pr(L)$.

Proof : "$\Leftarrow$" It follows from Theorem 2. 1 .

"$\Rightarrow$" Obviously $A_0^x(x)=0$. It is sufficient to verify $A_0^x(y)=1$ for every $y(\neq x)$. Assume that $A_0^x(y)=a \neq 1$, then there exists $\eta \in pr(L)$ such that $1 \not\leq \eta$, $a \leq \eta$. Take $A_\eta^y \in \delta$. By T_2 , there exist B_1 , B_2 $\in \delta$ such that $B_1 \not\leq A_\eta^y$, $B_2 \not\leq A_\lambda^x$ and $B_1 \wedge B_2 = 0$, thus $B_1 \tilde{<} A_\lambda^x$. Since $B_1 \not\leq A_\eta^y$, $A_0^x(y) \not\leq \eta$. Therefore $A_0^x(y) \neq a$, a contradiction .

References

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