# The Weak-autocontinuity of Set Function and Its Applications

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Abstract: In this paper, The concept of weak-autocontinuity of set function is introduced. Relations between this concept and the concepts of autocontinuity and null-additivity and $\lambda$-subadditivity are given. We also obtain the conditions for Egoroff's Theorem and Riesz's Theorem. They are weaker than conditions given in [2,3] and one of them is the necessary condition for Riesz's Theorem.

Keywords: Autocontinuity, Set function; Weak-autocontinuity.

**1. Introduction**

Throughout this paper, let X be a classical nonempty set, **F** be a $\sigma$-algebra, $\{f_n\}$ and $f$ be measurable functions. All concepts and signs not defined in this paper may be found in [1,2,3,4].

In [2] the following concept is given:

Definition 1.1 A set function $\mu$ is called autocontinuous from above (resp. from below) if we have

$$\mu(A \cup B_n) \to \mu(A) \text{ (resp. } \mu(A - B_n) \to \mu(A))$$

whenever $A \in \mathbf{F}$, $B_n \in \mathbf{F}$, $A \cap B_n = \Phi$ (resp. $B_n \subset A$), n=1,2,3,..., $\mu(B_n) \to 0$.

$\mu$ is called autocontinuous if it is both autocontinuous from above and autocontinuous from below.

In [4], Zhao gives

Definition 1.2 A fuzzy measure $\mu$ in a fuzzy measure space is called $\lambda$-subadditive if whenever $A, B \in \mathbf{F}$, we have

$$\mu(A \cup B) \leqslant \lambda\mu(A) + \lambda\mu(B), \quad \lambda \geqslant 1.$$

In [2,3,4], Wang and Zhao give their own asymptotic structural characteristics respectively, and give the following results.

Theorem 1.3 (Egoroff's Theorem) If $\mu$ is autocontinuous from above(resp. $\lambda$-subadditive) and $A \in \mathbf{F}$, then

$$f_n \xrightarrow{a.e.} f \quad \text{is equivalent with} \quad f_n \xrightarrow{a.u.} f \text{ on A.}$$

Theorem 1.4 (Riesz's Theorem) Suppose $\mu$ is autocontinuous from above(resp. $\lambda$-subadditive) and $A \in \mathbf{F}$, If $f_n \xrightarrow{\mu} f$ on A, then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$, such that $f_{n_k} \xrightarrow{a.e.} f$.

Conclusions under the condition of autocontinuous from below can be found from [2]. Concepts of $\{f_n\}$ converges to $f$ " almost everywhere "(a.e.) and " pseudo-almost everywhere "(p.a.e.)," almost uniform "(a.u.) on A come from [1,2,3] wholely.

**2. Concept of Weak-autocontinuity**

Definition 2.1 Let $\mu$ be a set function, if for any $E_n$, $A \in \mathbf{F}$, n=1,2,..., and $\mu(E_n) \to 0$, there exists a subsequence $\{E_{n_k}\}$ of $\{E_n\}$, such that

$$\mu(A\cup(\bigcup_{k=m}^{\infty} E_{n_k}))\to\mu(A) \text{ (resp. } \mu(A-(\bigcup_{k=m}^{\infty} E_{n_k}))\to\mu(A)),$$

then $\mu$ is called weak-autocontinuous from above (resp. weak-autocontinuous from below).

$\mu$ is called weak-autocontinuous if it is both weak-autocontinuous from above and weak-autocontinuous from below.

Theorem 2.2 For any $E_n\in\mathbf{F}$, n=1,2,3,..., $\mu(E_n)\to 0$, if there exists $\{\varepsilon_k\}$, where $\{\varepsilon_k\}$ is a sequence of subsequences of $\{E_n\}$ : $\varepsilon_k=\{E_{n_i}^{(k)}\}$, k=1,2,3,...,

such that $\lim\limits_{k\to\infty}\mu(A\cup(\bigcup_{i=1}^{\infty} E_{n_i}^{(k)}))=\mu(A)$, then $\mu$ is weak autocontinuous from above

(resp. $\lim\limits_{k\to\infty}\mu(A-\bigcup_{i=1}^{\infty} E_{n_i}^{(k)})=\mu(A)$, then $\mu$ is weak-autocontinuous from below).

Proof. From the condition of the theorem, we know for $E_n\in\mathbf{F}$, n=1,2,3,..., $\mu(E_n)\to 0$, there exists $\{\varepsilon_k\}$, where $\varepsilon_k=\{E_{n_i}^{(k)}\}$, such that

$$\lim_{k\to\infty}\mu(A\cup(\bigcup_{i=1}^{\infty} E_{n_i}^{(k)}))=\mu(A).$$

So first, we can obtain $\varepsilon_1=\{E_{n_i}^{(1)}\}$, such that $\mu(A\cup(\bigcup_{i=1}^{\infty} E_{n_i}^{(1)}))<\mu(A)+1$.

For this $\varepsilon_1$, because $\lim\limits_{i\to\infty}\mu(E_{n_i}^{(1)})=0$ is true, so furthermore there exists a

subsequence $\varepsilon_2=\{E_{n_i}^{(2)}\}$ of $\varepsilon_1$, such that $\mu(A\cup(\bigcup_{i=1}^{\infty} E_{n_i}^{(2)}))<\mu(A)+\frac{1}{2}$. In general,

there exists a subsequence $\varepsilon_k=\{E_{n_i}^{(k)}\}$ of $\{E_{n_i}^{(k-1)}\}$, such that

$$\mu(A\cup(\bigcup_{i=1}^{\infty}E_{n_i}^{(k)}))<\mu(A)+\frac{1}{k},\ k=2,3,\ldots.$$

If we take $n_i=n_i^{(i)}$, then $\{E_{n_i}\}$ is a subsequence of $\{E_n\}$ and $\bigcup_{i=1}^{\infty}E_{n_i}\subset\bigcup_{i=1}^{\infty}E_{n_i}^{(k)}$, $k=1,2,\ldots$,

Consequently,

$$\mu(A\cup(\bigcap_{k=1}^{\infty}\bigcup_{i=k}^{\infty}E_{n_i}))\leqslant\mu(A\cup(\bigcup_{i=k}^{\infty}E_{n_i}))\leqslant\mu(A\cup(\bigcup_{i=1}^{\infty}E_{n_i}^{(k)}))\leqslant\mu(A)+\frac{1}{k}$$

for all $k=1,2,3,\ldots$, and therefore $\lim_{k\to\infty}\mu(A\cup(\bigcup_{i=k}^{\infty}E_{n_i}))=\mu(A)$.

The proof for the situation of weak-autocontinuous from below is similar.

The following proposition gives the relations between weak-autocontinuity from above with other concepts.

Proposition 2.3

1). If $\mu$ is a possibility measure, then $\mu$ is weak-autocontinuous from above;
2). If $\mu$ is autocontinuous from above, then $\mu$ is weak-autocontinuous from above;
3). If $\mu$ is a quasi-measure, then $\mu$ is weak-autocontinuous from above;
4). If $\mu$ is a $\lambda$-fuzzy measure, then $\mu$ is weak-autocontinuous from above;
5). If a fuzzy measure $\mu$ is $\lambda$-subadditive, then $\mu$ is weak-autocontinuous from above at 0.

Proof.

1). Suppose $\mu$ is a possibility measure, then $\mu(\bigcup_{i\in I}A_i)=\sup_{i\in I}\mu(A_i)$.

So if $E_n\in\mathbf{F}$, $\mu(E_n)\to 0$, $n=1,2,3,\ldots$, then for any $\varepsilon>0$, there exists a natural number N such that $\mu(E_N)<\varepsilon$, $(n>N)$. Now we take $E_{n_1}=E_N$, $E_{n_2}=E_{N+1}$ ,...,

then $\mu(A\cup(\bigcup_{k=1}^{\infty}E_{n_k}))<\mu(A)+\varepsilon$. From Theorem 2.2 we know $\mu$ is weak-autocontinuous from above.

2). If $\mu$ is autocontinuous from above, then $\mu(A\cup E_n)\to\mu(A)$ for any $\{E_n\}$ such that $\mu(E_n)\to 0$. So for any $\varepsilon>0$, there exists $E_{n_1}$ such that

$\mu(A\cup E_{n_1})<\mu(A)+\frac{\varepsilon}{2}$, furthermore, from $\mu(A\cup E_{n_1}\cup E_n)\to\mu(A\cup E_{n_1})$ we know there exists $n_2>n_1$ such that $\mu(A\cup E_{n_1}\cup E_{n_2})<\mu(A\cup E_{n_1})+\frac{\varepsilon}{2^2}<\mu(A)+\frac{3\varepsilon}{4}$, In general, we take $n_{k+1} > n_k$ such that

$$\mu(A\cup(\bigcup_{i=1}^{k+1} E_{n_i})) = \mu(A\cup(\bigcup_{i=1}^{k} E_{n_i})\cup E_{n_{k+1}}) < \mu(A)+(1-\frac{1}{2^{k+1}})\varepsilon < \mu(A)+\varepsilon .$$

Now we obtain a subsequence $\{E_{n_i}\}$ of $\{E_n\}$ such that $\mu(A\cup(\bigcup_{i=1}^{\infty} E_{n_i})) < \mu(A)+\varepsilon$ .

From Theorem 2.2 we know that $\mu$ is weak-autocontinuous from above. Similarly, if $\mu$ is autocontinuous from below, we can easily obtain the corresponding conclusion.

3). A quasi-measure is autocontinuous from above has been proved in [1].

4). Also in [1], it has been proved that a $\lambda$-fuzzy measure is a quasi-measure.

5). Suppose $\mu$ is $\lambda$-subadditive, for any $E_n \in \mathbf{F}$, n=1,2,3,..., and $\mu(E_n)\to 0$, we take $n_1 < n_2 < \cdots < n_k < \cdots$, such that

$$\mu(E_{n_1}) < \frac{\varepsilon}{2\lambda},\ \mu(E_{n_2}) < \frac{\varepsilon}{(2\lambda)^2},\ \ldots\ldots,\ \mu(E_{n_k}) < \frac{\varepsilon}{(2\lambda)^k}, \ldots .$$

Because $\mu$ is $\lambda$-subadditive, we have

$$\mu(\bigcup_{i=1}^{k} E_{n_i}) \leqslant \lambda\,\mu(E_{n_1}) + \lambda^2\,\mu(E_{n_2}) + \ldots + \lambda^k\,\mu(E_{n_k}) < (1-\frac{1}{2^k})\varepsilon < \varepsilon .$$

Let $k\to\infty$, then $\mu(\bigcup_{k=1}^{\infty} E_{n_k}) < \varepsilon$, from Theorem 2.2 we know that $\mu$ is weak- autocontinuous from above at 0 . ■

The following example shows that although weak-autocontinuity from above at 0 can be obtained from autocontinuity from above or $\lambda$-subadditivity, the former is actually weaker than the laters.

Example 2.4 Let ,X={1,2,3,...}, **F=P**(X).

$$\mu(E) = (\mathrm{Car}E)\sum_{i\in E}\frac{1}{2^i},\quad E\in\mathbf{F}.$$

We easily know that $\mu$ is a fuzzy measure and it is weak-autocontinuous from above at 0.

In fact, for any $B_n \in \mathbf{F}$, n=1,2,3,..., and $\mu(B_n)\to 0$, if there exists $m\in\overline{\lim_{n\to\infty}} B_n$, then there exists $B_{n_1}$, $B_{n_2}$, ..., $B_{n_k}$, ..., such that $m\in B_{n_k}$, k=1,2,3,.... From the definition of $\mu$, we know that $\mu(B_{n_k}) \geqslant \frac{1}{2^m}$ k=1,2,3,....

So $\mu(B_n) \not\to 0$. It's a contradiction. That is to say $\overline{\lim_{n\to\infty}} B_n = \Phi$.

But $\mu$ is not autocontinuous from above, this result has been given in [1].

Now we show that $\mu$ is not $\lambda$-subadditive.

Suppose there exists $\lambda_0 > 1$, such that $\mu(A\cup B) \leqslant \lambda_0(\mu(A)+\mu(B))$ for arbitrary $A,B \in \mathbf{F}$. Now we take $E_1=\{1\}, E_2=\{m_1\}, E_3=\{m_2\},\ldots, E_{n+1}=\{m_n\}$, where

$$\frac{\lambda_0}{2^{m_1}} < \frac{1}{2},\ \frac{\lambda_0^2}{2^{m_2}} < \frac{1}{2^2},\ldots,\ \frac{\lambda_0^n}{2^{m_n}} < \frac{1}{2^n}.$$

So we have $$\mu(\bigcup_{i=1}^{n+1} E_i) = \mu(1,m_1,m_2,\ldots,m_n) = (n+1)(\frac{1}{2}+\frac{1}{2^{m_1}}+\ldots+\frac{1}{2^{m_n}}) > \frac{n+1}{2}.$$

On the other hand, if $\mu$ is $\lambda$-subadditive, we should have

$$\mu(\bigcup_{i=1}^{n+1} E_i) \leqslant \lambda_0\,\mu(E_1) + \lambda_0^2\,\mu(E_2)+\ldots+\lambda_0^{n+1}\,\mu(E_{n+1}) = \frac{\lambda_0}{2}+\frac{\lambda_0^2}{2^{m_1}}+\ldots+\frac{\lambda_0^{n+1}}{2^{m_n}}.$$

$$< \lambda_0(\frac{1}{2}+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n}) = \lambda_0(\frac{1}{2}+1-\frac{1}{2^n}) < \frac{3}{2}\lambda_0.$$

When we take $n > 3\lambda_0 - 1$, we will have a contradiction. So $\mu$ is not $\lambda$-subadditive.

Theorem 2.5 If $\mu$ is weak-autocontinuous from above, then it is null-additive.

Proof. It is very evident when we take $E_n \equiv \Phi$, $n=1,2,3,\ldots$.

The following example shows that null-additivity is weaker than weak-autocontinuity from above.

Example 2.6 Let $X=\{0,1,2,\ldots\}$, $\mathbf{F}=\mathbf{P}(X)$ and

$$\mu(E) = \begin{cases} \sum_{i\in E} \frac{1}{2^{i+1}}, & 0 \notin E. \\ \infty, & 0\in E \text{ and } E-\{0\}\neq\Phi. \\ 1, & E=\{0\}. \end{cases}$$

Then $\mu$ is a fuzzy measure and it is null-additive (See [1]). But now we show that it is not weak-autocontinuous from above.

In fact, if we take $A=\{0\}$, $E_n=\{n\}$, $n=1,2,3,\ldots$, then $\mu(E_n)\to 0$. Suppose $\{E_{n_k}\}$ is an arbitrary subsequence of $\{E_n\}$, from the definition of $\mu$ we know

$$\mu(A\cup(\bigcup_{k=m}^{\infty} E_{n_k})) \equiv \infty, \text{ but } \mu(A)=1.$$

This example shows that the concept of weak-autocontinuity from above is a concept between autocontinuity and null-additivity.

### 3. Convergence Theorems

Theorem 3.1(Egoroff's Theorem). Suppose $A \in \mathbf{F}$, $\mu(A) < \infty$, and $\mu$ is weak-autocontinuous from above at 0. If

$$f_n \xrightarrow{a.e.} f \text{ on A, then } f_n \xrightarrow{a.u.} f \text{ on A.}$$

Proof. Write $E_n^m = \bigcap_{i=n}^{\infty} \{x \mid |f_i - f| < \frac{1}{m}\}$, m=1,2,3,..., then $E_1^m \subset E_2^m \subset \ldots$ and

$\{x \mid f_n \to f\} = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} E_n^m$. Let B={x | $f_n \not\to f$}, because $f_n \xrightarrow{a.e.} f$ on A, so $\mu(B)=0$,

and $\bigcup_{n=1}^{\infty} E_n^m \supset$ A-B, m=1,2,3,.... Thus $\lim_{n\to\infty}(A - E_n^m) = A - \bigcup_{n=1}^{\infty} E_n^m \subset B$. From the monotoneity of $\mu$

we have $0 \leq \lim_{m\to\infty} \mu(\lim_{n\to\infty}(A - E_n^m)) \leq \mu(B) = 0$.

Furthermore, from the continuity of $\mu$ and the condition $\mu(A) < \infty$, we can know that $\lim_{n\to\infty} \mu(A - E_n^m) = 0$ is true for every m=1,2,3,....

So for every m, there exists $n_m$ such that $\mu(A - E_{n_m}^m) < \frac{1}{m}$. If we note $F_m = A - E_{n_m}^m$,

then $\lim_{m\to\infty} \mu(F_m) = 0$. Because $\mu$ is weak-autocontinuous from above at 0, there exists a

subsequence $\{F_{m_i}\}$ of $\{F_m\}$ such that $\mu(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} F_{m_i}) = \mu(\Phi) = 0$. Therefore,

for every $\varepsilon > 0$, there exists a subsequence (It's no harm to note it $\{F_{m_i}\}$ again)

such that $\mu(\bigcup_{i=1}^{\infty} F_{m_i}) < \varepsilon$. Then $\{f_n\}$ converges to $f$ uniformly on $A - \bigcup_{i=1}^{\infty} F_{m_i}$.

In fact, for any $\varepsilon'>0$, we take $i_0$ satisfying $m_{i_0} > \frac{1}{\varepsilon'}$, and we note $k=m_{i_0}$,

then for every $x \in A - \bigcup_{i=1}^{\infty} F_{m_i}$, it satifies $x \in A$ and $x \notin F_k$, therefore $x \in E^k_{n_k}$, that is to say

$x \in \bigcap_{i=n_k}^{\infty} \{ | f_i - f | < \frac{1}{k} \}$. So when we take $i \geq n_k$, it must have $| f_i - f | < \frac{1}{k} < \varepsilon'$. ■

For the case of weak-autocontinuous from below we have the following result:

Theorem 3.2 Suppose $A \in \mathbf{F}$, $\mu(A) < \infty$, and $\mu$ is weak-autocontinuous from below,

if $f_n \xrightarrow{a.e.} f$ on A, then $f_n \xrightarrow{p.a.e.} f$ on A.

Theorem 3.3 (Riesz's Theorem) Suppose $A \in \mathbf{F}$, then $f_n \xrightarrow{\mu.} f$ on A implies there

exists a subsequence $\{ f_{n_k} \}$ of $\{ f_n \}$ such that $f_{n_k} \xrightarrow{a.e.} f$ if and only if $\mu$ is weak-autocontinuous from above at 0.

Proof.

Sufficiency. There is no harm in assuming A=X. For any $\{ f_n \}$ and $f$, if $f_n \xrightarrow{\mu.} f$ on A, then for every k=1,2,3,…, there exists $n_k$ respectively, such that

$$\mu(\{x \mid | f_{n_k}(x) - f(x) | \geq \frac{1}{k}\}) < \frac{1}{k}, \quad k=1,2,3,\ldots.$$

Without any loss of generality, we suppose $n_{k+1} > n_k$, k=1,2,3,…. If we note

$E_k = \{x \mid | f_{n_k}(x) - f(x) | \geq \frac{1}{k}\}$, then $\lim_{k\to\infty} \mu(E_k) = 0$. As $\mu$ is weak-autocontinuous

from above at 0, there exists a subsequence $\{ E_{k_i} \}$ of $\{ E_k \}$, such that

$$\mu\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} E_{k_i}\right) = 0. \text{ Now we prove that } f_{n_k} \longrightarrow f \text{ on } X - \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} E_{k_i}.$$

If $x \in X - \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} E_{k_i}$, then $x \in \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} E^c_{k_i}$. So surely there exists j(x) such that

$x \in \bigcap_{i=j(x)}^{\infty} E_{k_i}^c$, that is to say $| f_{n_{k_i}} - f | < \frac{1}{k_i}$ when $i \geqslant j(x)$. This conclusion means

we have proved the following result:

For every given $\varepsilon > 0$, we first take $i_0$ such that $\frac{1}{k_{i_0}} < \varepsilon$, then we take $i \geqslant j(x) \vee i_0$, now we have

$$| f_{n_{k_i}} - f | < \frac{1}{k_i} \leqslant \frac{1}{k_{i_0}} < \varepsilon$$

and $\{ x \mid f_{n_{k_i}} \not\to f \} \subset \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} E_{k_i}$. So, $f_{n_{k_i}} \xrightarrow{a.e.} f$.

Necessity. Suppose $E_n \in \mathbf{F}$, n=1,2,3,..., and $\mu(E_n) \to 0$. We want to prove that there exists a subsequence $\{E_{n_i}\}$ of $\{E_n\}$ such that $\mu(\bigcap_{m=1}^{\infty} \bigcup_{i=m}^{\infty} E_{n_i})=0$.

Let $f_n(x) = \begin{cases} 1 & x \in E_n, \\ 0 & x \notin E_n, \end{cases}$ and $f(x) \equiv 0$.

Evidently, $f_n \xrightarrow{\mu.} f$. From the hypothesis of the theorem, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \xrightarrow{a.e.} f$. That means $\mu(\{x \mid f_{n_k} \not\to f\})=0$.

Note $\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E_{n_k} \subset \{x \mid f_{n_k} \not\to f\}$, the conclusion follows. ■

In paper [2] and [4], properties of measurable functions were discussed by using the properties of $\mu$. But the necessity of theorem 3.3 tells us that we can use the properties of measurable functions to discuss the properties of $\mu$.

For the case of weak-autocontinuous from below, we have the following result:

Theorem 3.4 Suppose $A \in \mathbf{F}$, then $f_n \xrightarrow{\mu.} f$ on A implies there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \xrightarrow{p.a.e.} f$ if and only if $\mu$ is weak-autocontinuous from below.

The proof is similar to Theorem 3.3.

Until now, we get weaker conditions than [2,4] to satisfy Egoroff's theorem and Riesz's theorem.

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