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UTILISATION OF THE FUZZY RELATION AND α-CUT OF FUZZY SET FOR MAKING GROUP DECISIONS

1. Preferential orderings of decision makers

We consider the situation in which there is a group of decision makers, each of whom has an opportunity for making a certain number of decisions. Each member of the group has his preferential decisions. The task is how to obtain a certain preferential ordering appropriate for the whole group, on the basis of the preferential ordering of each member of the group. We utilise the method based on the theory of fuzzy sets proposed by Blin [1] and Blin and Whinston [2], which are also discussed in this work [6].We assume the following model of decision making with the data:

$A = \{a_1,\ldots,a_m\}$ - a set of decisions,

$B = \{b_1,\ldots,b_n\}$ - a set of decision makers (members of the group),

$O_k \subset A \times A$ - an unfuzzy ordering preferential for k-decision maker.

If a decision maker prefers a_i to a_j, a pair of decisions $(a_i, a_j) \in O_k$, which is written down as $a_i > a_j$.

Preferential orderings of decisions of different decision makers can be different or even contradictory to each other. The problem to be solved is to find a preferential ordering of decisions which would be the most suitable for and characteristic of all the group. Generally speaking, the task of making group decisions is to determine a certain transformation:

$$(O_1, O_2, \ldots, O_n) \rightarrow O_0. \qquad (1)$$

On the basis of preferential orderings of individual members of the group we determine the most adequate group preferential ordering O_0.

***Definition 1** [6]*

A group preferential ordering is called a social preference and is defined as a fuzzy relation $R \subset A \times A$ expressed by membership function:

$$\mu_R : A \times A \to [0,1]. \tag{2}$$

The values of membership function $\mu_R(a_i, a_j)$ determine the degree of the preference of decision a_i over decision a_j. Fuzziness in determining a preferential ordering of decisions for the whole group is expressed here by membership function μ_R, the values of which are fractions from interval [0,1]. For example, if all the decision makers think that decision „a" is preferable to decision „b", we can say that social (group) preference is such that a>b. We assume then that:

$$\mu_R(a, b) = 1. \tag{3}$$

However, if some decision makers prefer decision „a" while the others decision „b", the degree of the group preference of „a" over „b" can amount to e.g. 0.7,0.6 or 0.3. We can say that decision „a" is preferred to decision „b" only to some extent. If we deal with the set of n decision makers and m decisions, it becomes more complicated to determine a preferential ordering of all the decisions.

2. Determining of group social preference

For each decision maker b_k a matrix of preferences S_k on the basis of the set of preferential ordering O_k is created.

$$S_k = \left[s_{ij}^k\right], \text{ where } s_{ij}^k = \begin{cases} 1, & \text{when } (a_i, a_j) \in O_k, \\ 0, & \text{when } (a_i, a_j) \notin O_k. \end{cases} \tag{4}$$

On the basis of the matrix of preferences S_k we form a total matrix N of individual preferences:

$$N = \sum_{k=1}^{n} S_k. \tag{5}$$

As a social preference of the whole group of decision makers we assume a fuzzy relation with values of membership:

$$\mu_R(a_i, a_j) = \frac{1}{n} n_{ij}, \tag{6}$$

where n_{ij} are the elements of matrix N.

Our task is to determine a certain unfuzzy preferential ordering of decisions on the basis of this fuzzy relation reflecting a social preference of the group. The method of determining a group preference is based on the concept of α-cut of a fuzzy set.

First we define α-cut of fuzzy relation R for parameter $\alpha=\tau$, where τ is a certain level of acceptance of a preference in the group.

$$R_\tau = \{(a_i, a_j): \mu_R(a_i, a_j) \geq \tau\} \quad (7)$$

Set R_τ contains a pair of decisions included in a preferential ordering for which the level of acceptance is not lower than level τ accepted by the whole group. We mention here the method proposed by Blin [1], and described in the work [6].

1. First we have to order all the elements of matrix μ_R different from zero in strongly decreasing sequence $\tau_1, \tau_2, \ldots, \tau_s$. It means that if in matrix μ_R a few elements have the same value, that value will occur only once in the formed sequence.
2. We determine set R_{τ_1}
3. We check if the determined set allows us to determine a preferential ordering for all the decisions.

   Notice.

   A preferential ordering will be determined when set R_τ contains m(m-1)/2 pairs of decisions. Each decision has to occur in exactly m-1 pairs of set R_τ. The most preferred decision has to occur m-1 times in the first position in the pairs. The second most preferred decision has to occur m-2 times in the first position and once in the second and so on. The least preferred decision will occur m-1 times in the second position in the pairs but it will not occur in the first position
4. If it is not possible to define a preferential ordering, we determine a new set R_{τ_i} for the following i.
5. From that set we eliminate the pairs which show the preference contradictory to the pairs of set $R_{\tau_{i-1}}$.
6. Now we have to come back to point 3.
7. If we managed to determine a preferential ordering in point 4, it is the end of calculations. It is the wanted group preferential ordering.

8. If for i = 1,...,s we did not manage to determine a preferential ordering of decisions on the basis of the sets R_{τ}, we state that such a group preferential ordering does not exist. The opinions of the people making decisions were too divergent. They should discuss and specify the criteria to establish preferences and present their opinions once again, not so much different from each other.

Here is a simple example of the utilisation of the above procedure.

Example 1

A mine manager has appointed a group of ten people who are supposed to establish the importance and the order of making urgent decisions. The following symbols denote these decisions:

a - the purchase of a new combined cutter loader KWB-3RUW/4000 and walling PIOMA 25/45 OZ,

b - the purchase of new computers IBM PC and computer network UNIX WARE,

c - sending a group of employees to England on a modern management course,

d - the expansion of the warehouse.

The decision makers have presented the order of preferred decisions in the following form:

A decision maker	Preference of decision
b_1	c > a > b > d,
b_2, b_3	b > a > c > d,
b_4	c > d > b > a,
b_5, b_6, b_7	b > a > d > c,
b_8	d > c > a > b,
b_9, b_{10}	a > d > b > c.

The following preferential orderings of individual decision makers correspond to this notation:

$O_1 =$ {(c, a), (c, b), (c, d), (a, b), (a, d), (b, d)},

$O_2 = O_3 =$ {(b, a), (b, c), (b, d), (a, c), (a, d), (c, d)},

$O_4 =$ {(c, d), (c, b), (c, a), (d, b), (d, a), (b, a)},

$O_5 = O_6 = O_7 =$ {(b, a), (b, d), (b, c), (a, d), (a, c), (d, c)},

$O_8 =$ {(d, c), (d, a), (d, b), (c, a), (c, b), (a, b)},

$O_9 = O_{10} =$ {(a, d), (a, b), (a, c), (d, b), (d, c), (b, c)},

On the basis of these orderings we create the matrices of preferences S_k for each person making decisions:

$$S_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \quad S_2 = S_3 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array},$$

$$S_4 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}, \quad S_5 = S_6 = S_7 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array},$$

$$S_8 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}, \quad S_9 = S_{10} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

On the basis of these matrices we form a total matrix N:

$$N = \sum_{k=1}^{10} S_k = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{bmatrix} 0 & 4 & 7 & 8 \\ 6 & 0 & 7 & 6 \\ 3 & 3 & 0 & 4 \\ 2 & 4 & 6 & 0 \end{bmatrix} \end{array}.$$

Now we determine a social (group) preference as a fuzzy relation with the following membership function:

$$\mu_R = \frac{1}{10}\cdot N = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 0.4 & 0.7 & 0.8 \\ 0.6 & 0 & 0.7 & 0.6 \\ 0.3 & 0.3 & 0 & 0.4 \\ 0.2 & 0.4 & 0.6 & 0 \end{bmatrix} \end{array}.$$

We calculate relation R_τ by formula (7):

$R_{\tau=0.8}$ = {(a, d)},

$R_{\tau=0.7}$ = {(a, c), (a, d), (b, c)},

$R_{\tau=0.6}$ = {(a, c), (a, d), (b, a), (b, c), (b, d), (d, c)}.

We can stop calculations here because we have already received the ordering for $R_{\tau=0.6}$ containing all the pairs. Set $R_{\tau=0.6}$ is the wanted group preferential ordering O_0. On the basis of set O_0 we can write down the decisions in the order of their preference by the group in a convenient and clear way:

b>a>d>c.

It is the solution to the above problem of making group decisions.

These are the other R_τ:

$R_{\tau=0.4}$ = {(a, b), (a, c), (a, d), (b, a), (b, c), (b, d), (c, d), (d, b), (d, c)},

$R_{\tau=0.3}$ = {(a, b), (a, c), (a, d), (b, a), (b, c), (b, d), (c, a), (c, b), (c, d), (d, b), (d, c)},

$R_{\tau=0.2}$ = {(a, b), (a, c), (a, d), (b, a), (b, c), (b, d), (c, a), (c, b), (c, d), (d, a), (d, b), (d, c)}.

The most important decision which should be made first is the decision about the purchase of computers and computer network for the mine. Then a new combined cutter loader and walling should be bought. The next decision is to expand the warehouse and the last decision is to send employees on the course to England.

Here is another way of determining the membership function of fuzzy relation R representing a social preference [6]. Instead of formula (6) we can apply the following formula:

$$\mu_R(a_i, a_j) = r_{ij} = \begin{cases} (n_{ij} - n_{ji})/n, & \text{when } n_{ij} > n_{ji}, \\ 0, & \text{when } n_{ij} \le n_{ji}. \end{cases} \qquad (8)$$

Example 2

We determine now a group ordering for ten people making decisions in example 1 applying formula (8). We obtain membership function μ_R in the form:

$$\mu_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 0 & 0.4 & 0.6 \\ 0.2 & 0 & 0.4 & 0.2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \end{bmatrix} \end{array}.$$

We calculate the following R_τ by formula (7):

$R_{\tau=0.6} = \{(a, d)\}$,

$R_{\tau=0.4} = \{(a, c), (a, d), (b, c)\}$,

$R_{\tau=0.2} = \{(a, c), (a, d), (b, a), (b, c), (b, d), (d, c)\}$.

We have obtained the ordering which is the same as the previous one: b>a>d>c.

To sum up we can state that the presented method can be utilised in different fields of life, not only in mining. It is useful in situations in which we are supposed to make a series of decisions and the importance of decisions is estimated by a group of experts. The preferences of individual experts for some decisions can be different or even contradictory. The presented method allows us to determine a social (group) preference for the whole team of the people making decisions. However, it is not always possible to determine such a preference for the whole group. In case of a big divergence of opinions such a group preference does not exist. Let's consider the simplest example. Let experts b_1 and b_2 have two decisions a_1 and a_2 to choose from. The expert b_1 prefers decision a_1 while the expert b_2 prefers decision a_2. We deal with the following preferential orderings:

$$O_1 = \{(a_1, a_2)\}, \qquad O_2 = \{(a_2, a_1)\}.$$

These orderings are contradictory to each other. In such a situation the experts should discuss their opinions in order to agree upon their preferences or to appoint another expert. The same concerns the situation in which there are more experts and decisions. If their opinions are too divergent, they should discuss them and specify the criteria to establish preferences and present their positions once again.

Literature

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Abstract

The paper presents α method of making group decisions on the basis of preferential ordered sets of individual members of the group. The theory of fuzzy sets has been used for determining group decisions. The method of determining group decisions on the basis of this theory has been proposed in the works of Blin [1] and Blin and Whinston [2]. In this method it is assumed that the following data are given:

$A = \{a_1, \dots, a_m\}$ - a set of decisions,

$B = \{b_1, \dots, b_n\}$ - a set of decision makers (members of the group).

$O_k \subset A \times A$ - an unfuzzy ordering preferential for k-decision maker.

Preferential orderings of decisions of two different decision makers can be different or even contradictory to each other. The problem to be solved is to find a preferential ordering of decisions which would be the most characteristic of and suitable for all the group.

A group preferential ordering is called a social preference and is defined as fuzzy relation $R \subset A \times A$ expressed by membership function $\mu_R : A \times A \to [0,1]$. The values of membership function $\mu_R(a_i, a_j)$ determine the degree of the preference of decision a_i over decision a_j. Fuzziness in determining a preferential ordered set of

decisions for all the group is expressed here by membership function μ_R, the values of which are fractions from the interval [0, 1].

The paper presents and makes use of only a few notions from the theory of fuzzy sets. The following notions have been discussed:

- a definition of a fuzzy set,
- a fuzzy set cut,
- a definition of a fuzzy relation.

In order to acquaint oneself with the theory of fuzzy sets in a more precise way, professional literature which has been given should be referred to [3], [4], [5].

The method of determining a group social preference has been presented too. In this case the notion of α-cut of a fuzzy set has been used. At the end of the paper there is an example of making a group decision by ten decision makers having four decisions at their disposal.