On n-reverse of implicators

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Abstract. For given implicator I we can construct the binary operation $I^*:[0;1]^2 \to [0;1]$ such that $I^*(x,y) = \max[0, \min(1, n(x) - n(y) + I(n(x), n(y)))]$, where n is an negator. The binary operation I* is called the reverse of implicator I. Generally, I* cann't be an implicator. The implicator I is reversible if I* is an implicator too. The condition of reversibility of implicators are studied.

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Preliminaries

Definition 1. The unary operator $n : [0;1] \to [0;1]$ is called *negator* iff

(n1) $a \le b \Rightarrow n(a) \ge n(b)$

(n2) $n(0) = 1, n(1) = 0$

The negator I is called **involutive** if $n(n(a)) = a$ for any $a \in [0,1]$

Definition 2. The binary operation $I: [0;1]^2 \to [0;1]$ is called *implicator* if for any $a,b,c \in [0;1]$ hold

(i1) $b \le c \Rightarrow I(a,b) \le I(a,c),\ I(b,a) \ge I(c,a)$ (hybrid monotonicity)

(i2) $I(0,0) = I(1,1) = 1,\ I(1,0) = 0$ (bondary conditions)

Corollary 1. Using (i1), (i2) we have: $1 = I(0,y) = I(x,1)$ for all $x,y \in [0,1]$

Definition 3. The implicator I is **contrapositive** with respect to negator n if

$$I(x,y) = I((n(y), n(x)) \text{ for all } x,y \in [0,1]$$

Definition 4. The implicator I is **border** if $I(1,y) = y$ for all $y \in [0,1]$.

There exist many ways how to construct implicators. One of them use t-norms and t-conorms.

Definition 5. The binary operation $T : [0;1]^2 \to [0;1]$ is called **t-norm** if for any $a,b,c \in [0;1]$ hold

(t1) $T(1,a) = a$ (boundary condition)

(t2) $a \le b \Rightarrow T(a,c) \le T(b,c)$ (monotonicity)

(t3) $T(a,b) = T(b,a)$ (commutativity)

(t4) $T(T(a,b),c)) = T(a,T(b,c))$ (associativity)

The binary operation $S : [0;1]^2 \to [0;1]$ defined by

$$S(x,y) = n(T(n(x), n(y)))$$

where T is a t-norm and n is an involutive negator, is called **t-conorm** dual to the t-norm T with respect to the negator n. Evidently then $S(x,y) = n(T(n(x), n(y)))$. If $n(x) = 1\text{-}x$ we shall simly say that S is dual to T.

Example 1. The family of Frank t-norms $\{T_s^F, s \in [0,\infty]\}$ is defined by

$T_0^F(x,y) = T_M(x,y) = \min(x,y)$ $T_1^F(x,y) = T_P(x,y) = xy$

$T_\infty^F(x,y) = T_L(x,y) = \max(0, x+y-1)$

$T_s^F(x,y) = \log_s\left(1+(s^x-1)(s^y-1)/(s-1)\right), s>0, s \neq 1$

The family of Frank t-conorms is given by $S_s^F(x,y) = 1 - T_s^F(1-x, 1-y)$

Let T be a t-norm S be a dual t-conorm , n be a negator then the binary operation

$$I(x,y) = n(T(x,n(y)) = S(n(x),y)$$

is implicator. If n is an involutive negator then such implicator I is so-called **model implicator** i.e. contrapositive border and fullfils the exchange principle:

$$I(a,I(b,c)) = I(b,I(a,c))$$

Example 2. Let $T(x,y) = T_P(x,y) = xy$ and $n(x) = 1 - x$. We obtain the Reichenbach implicator: $I_P(x,y) = 1 - x(1-y) = 1 - x - xy$

Example 3. Let $T(x,y) = T_L(x,y) = Max(0, x+y - 1)$, $n(x) = 1 - x$. We obtain the Lukasiewicz implicator $I_L(x,y) = Min(1,1-x+y)$

Example 4. $T(x,y) = T_M(x,y) = Min(x,y)$ and $n(x) = 1 - x$. We obtain the Kleene-Dienes implicator $I_M(x,y) = Max(1-x,y)$

Now we shall show some way how to generate implicators from given ones.

The reverse of implicators

Definition 6. Let I be an implicator, n be a negator. The binary operation $I_n^*: [0,1]^2 \to [0,1]$ defined by

$$I_n^*(x,y) = /n(x) - n(y) + I(n(x),n(y))/$$

where $/a/ = \min(1,\max(0,a))$, is called the **n-reverse of implicator I**.

Evidently, $I_n^*(0,0) = I_n^*(1,1) = I_n^*(0,1) = 1, I_n^*(1,0) = 0$, $I_n^*(x,0) = n(x), I_n^*(1,y) = 1 - n(y)$,

but I* need not be an implicator . If I_n^* is an implicator then we shall say that I is **n-reversible.**

Example 5. Let $I(x,y) = \begin{cases} 1 & \text{if } x=0 \text{ or } y=1 \\ 0 & \text{otherw.} \end{cases}$, n be a continuous negator. Take $y \in (0,1)$ such that $n(y) \in (0,1)$. Then $I_n^*(0,y) = /1-n(y)+I(1,n(y))/ = 1-n(y) \neq 1$ which contradicts (i1) . Thus I_n^* isn't implicator.

Theorem 1. If n is continuous negator and I_n^* is an implicator then $I(1,y) \geq y$.

Proof. If I_n^* is an implicator then $1=I_n^*(0,y) = /1-n(y)+I(1,n(y))/$. It follows that $1-n(y)+I(1,n(y)) \geq 1$ or $I(1,n(y)) \geq n(y)$. The continuity of n gives the required inequality.

Let n_1 be another negator. Then we define Lukasiewics n_1-implicator as follows:

$$I_{n_1,L}(x,y) = \min(1, n_1(x) + y)$$

Theorem 2. Let n be a continuous negator. Then $I_{n_1,L}$ is n-reversible iff $n_1(y) \geq 1-y$ for all $y \in [0,1]$.

Proof. $\left(I_{n_1,L}\right)_n^* = /n(x) - n(y) + \min\left(1, n_1\left(n(x)\right) + n(y)\right)/$

If $\left(I_{n_1,L}\right)_n^*$ is an implicator then $1 = \left(I_{n_1,L}\right)_n^*(x,1) = /n(x) + n_1\left(n(x)\right)/$.

Thus $n(x) + n_1\left(n(x)\right) \geq 1$. The continuity of n gives $y + n_1(y) \geq 1$ Conversly, let $y + n_1(y) \geq 1$.

Then $\left(I_{n_1,L}\right)_n^*(y,y) = /\min\left(1, n_1\left(n(y)\right) + n(y)\right)/ = 1$. It can be easilly shown that

$$\left(I_{n_1,L}\right)_n^* = 1 \quad \text{for} \quad y > x$$

and

$$\left(I_{n_1,L}\right)_n^*(x,y) = /n(x) - n(y) + 1/ \quad \text{for} \quad y \geq x$$

Such binary operation is the implicator.

Example 6. Let $n(x) = 1 - x, \quad n_1(x) = 1 - \sqrt{2x - x^2}$. Then

$$\left(I_{n_1,L}\right)_n^* = /y - x + \min(1, 2 - y - \sqrt{1 - x^2})/$$

isn't implicator. E.g. $\left(I_{n_1,L}\right)_n^*(0.5;1) = 0.63$

In the following we shall be using the negator n(x) = 1-x only. For such negator we shall denote $I^* = I_n^*$. It means

$$I^*(x,y) = /y - x + I(1 - x, 1 - y)/$$

If I^* is the negator then we say that I is reversible.

Theorem 3. If I is reversible implicator then I^* is an border implicator.

Proof. $I^*(1,y) = /y - 1 + I(0, 1 - y)/ = y$.

Then Theorem 3 allows to generate border implicators from another ones.

Theorem 4. If I is a contrapositive (with respect to n(x)=1-x) reversible negator The I^* is contrapositive negator.

Proof. $I^*(1 - y, 1 - x) = /y - x + I(y,x)/ = /y - x + I(1 - x, 1 - y)/ = I^*(x,y)$

Example 7. Let $I(x,y) = I_L(x,y) = \min(1, 1 - x + y)$ (Lukasiewics implicator) Then $I^* = I_L$.

Theorem 5. If $I \geq I_L$ then I is reversible and $I^* = I_L$

Proof. If $x \leq y$ then $1 - x \geq 1 - y$ and

$$I^*(x,y) = /y - x + I(1 - x, 1 - y)/ \geq /y - x + I_L(1 - x, 1 - y)/ = 1 = I_L(x,y)$$

If $x > y$ then $1 - x < 1 - y$ and

$$I^*(x,y) = /y - x + I(1 - x, 1 - y)/ \geq /y - x + 1/ = I_L(x,y)$$

Theorem 6. If I is a reversible implicator then $I(x,y) \geq I_M(x,y) = \mathrm{Max}(1 - x, y)$.

Proof. If I is reversible then $1 = I^*(x,1) = /1 - x + I(1 - x, 0)/$. Thus $I(1 - x, 0) \geq x$ or $I(x,0) \geq 1 - x$. Similarly, $1 = I^*(0,y) = /y + I(1, 1 - y)/$. Therefore $I(1, 1 - y) \geq 1 - y$ or $I(1,y) \geq y$

The next theorem gives some necessary and sufficient condition for the reversibility of implicators (without proof)..

Theorem 7. Let I be an implicator. Denote
$Q(x,y) = \min(I(x,y), I_L(x,y)) = \min(1, I(x,y), 1-x+y)$.
Then I is reversible iff $|Q(x,y)-Q(x,z)| \le |z-y|$ and $|Q(x,y)-Q(z,y)| \le |x-z|$ and
$I(x,y) \ge I_M(x,y) = \mathrm{Max}(1-x,y)$

Selfreversibility

If $I = I^*$ then the implicator I is called **selfreversible**. Example 6 implies that Lukasiewics implicator I_L is selfreversibe. For given t-norm T we can define a binary operation
$T^*: [0,1]^2 \to [0,1]$ such that

$$T^* = /x+y-1+T(1-x,1-y)/ = \max(0, x+y-1+T(1-x,1-y))$$

If T^* is t-norm too then T is called the t-reversible t-norm. If $T = T^*$ then T is called selfreversible. The selreversibility of t-norm T allows to generate a selfreversible implicator.
Theorem 8. Let T be a selfreversible t-norm, S is dual t-norm then the implicator

$$I(x,y) = 1-T(x,1-y) = S(1-x,y)$$

is the selfreversible implicator.
Proof.
Remark. In [ ] was proved that t-norm T is selfreversible iff T is a Frank norm or a symmetrical ordinal sum of Frank t-norm. Recall that the ordinal sum of t-norm is the t-norm defined as follows:
Let $\{(a_k, b_k), k \in K\}$ be a family of pairwise disjoint subintervals of [0,1] and $\{T_k, k \in K\}$ be a family of t-norms different from T_M. Then The ordinal sum $\{a_k, b_k, T_k\}$ is the t-norm defined by

$$T(x,y) = a_k + (b_k - a_k)T\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right) \text{ if } x,y \in [a_k, b_k]$$

$T(x,y) = \min(x,y)$ otherwise
The ordinal sum $\{a_k, b_k, T_k\}$ is symmetrical if for any $i \in K$ there exists $j \in K$ such that
$a_i + b_j = 1$, $a_j + b_i = 1$ and $T_i = T_j$

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