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CARISTI TYPE FUZZY HYBRID FIXED POINT THEOREM IN MENGER PROBABILISTIC METRIC SPACE

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ABSTRACT: In this paper, we bring forward the concept of Caristi type fuzzy hybrid fixed point in M-PM-space, gave a fuzzy hybrid fixed point theorem and a common fuzzy hybrid fixed point theorem of sequences of fuzzy mappings, our theorems improve and generalize the Caristi's fixed point theorem and correspoinding recant important results.

KEY WORDS: Probabilistic metric space, Menger space, Caristi's fixed point theorem, fuzzy hybrid fixed point, common fuzzy hybrid fixed point.

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1 INTRODUCTION AND PRELIMINARIES

IN 1976, Caristi proved the Caristi's fixed point theorem [1], because this theorem does not require the continuity of the mapping, it finds applications in many fields.

In 1991, Chang S. S. etc, bring forward Caristi's fixed point theorem and Ekeland's variational principle in PM-spaces. In 1993, Chang S. S. etc [4], bring forward set-valued Caristi's theorem in PM-spaces.

In 1997, Shi C. bring forward Caristi's type hybrid fixed point theorems in PM-spaces. In this paper, we bring forward the concept of Caristi type fuzzy hybrid fixed point in Menger-PM-space, gave a fuzzy hybrid fixed point theorem and a comonon fuzzy hybrid fixed point theorem of sequences of fuzzy mappings, our theorems improve and generalize the Caristi's fixed point and corresponding recent important results.

Throughout this paper, we assume the (E, F, Δ) is a Menger probabilistic metric space (briefly M-PM-space), where t-norm Δ satisfies the condition:

$$\lim_{x\to 1^{-1}} \Delta(x,y) = y, \forall y \in [0,1] \tag{1.1}$$

DEFINITION 1.1 a mapping $A: E \to [0,1]$ is called a fuzzy subset over E, we denote by $\mathscr{F}(E)$ the famity of all fuzzy subsets over E, a mapping $S: E \to \mathscr{F}(E)$ is called a fuzzy mapping over E. Let $S: E \to \mathscr{F}(E)$, $T: E \to E$, if $p \in E$ such that $Sp(p) = \max_{u\in E} Sp(u)$ and $Tp = p$, then p is called a fuzzy hybrid fixed point of S and T. Let $S_K: E \to \mathscr{F}(E)$ $(K = 1,2,\cdots\cdots)$, $T: E \to E$, , if $p \in E$ such that $(\bigcap_{K=1}^{+\infty} S_K p)(p) = \max_{u\in E}(\bigcap_{K=1}^{+\infty} S_K)(u)$ and $Tp = p$ then p is called a common fuzzy hybrid fixed point of $\{S_K\}$ and T.

LEMMA 1.1 (Chang S. S, etc [4],[8]) Let (E,F,Δ) be a complete M-PM-space, Δ satisfies the condition (1.1), $\Phi: E \to (-\infty, +\infty]$ be a lower semicontinuous function, bounded from below and $\not\equiv +\infty$, If for any $\varepsilon > 0$, $u \in E$ such that $\Phi(u) \leqslant \inf_{x\in E}\Phi(x) + \varepsilon$, then for every $\lambda > 0$, there exists some point $p \in E$ such that

(1) $F_{u,p}(t) \geqslant H(t - \frac{1}{\varepsilon\lambda}(\Phi(u) - \Phi(p))), \forall \lambda > 0$;

(2) $F_{u,p}(t) \geqslant H(t - \frac{1}{\lambda}), \forall \lambda > 0$;

(3) $\forall x \in E, x \neq p, \exists t_0 = t(x) > 0$ such that $F_{x,p}(t_0) < H(t_0 - \frac{1}{\varepsilon\lambda}(\Phi(p) - \Phi(x))$.

LEMMA 1.2 Let (E,F,Δ) and Δ satisfies the conditions of lemma 1.1, $T: E \to E$ satisfies: $\forall x,y \in E$

$$F_{x,Ty}(t) \leqslant F_{Tx,Ty}(t), \forall t > 0 \tag{1.2}$$

then $T(E) = \{z \mid z = Tx, x \in E\} = F(E) = \{x \mid x = Tx, x \in E\}$

PROOF. It is obvious that $F(E) \subseteq T(E)$. $\forall z_1 \in T(E), \exists x_1 \in E, z_1 = Tx_1$, by (1.2).

$F_{Tz_1,Tx_1}(t) \geqslant F_{z_1,Tx_1}(t) = F_{Tx_1,Tx_1}(t) = H(t), \forall t \in R$, thus we have $Tz_1 = Tx_1 = z_1$, i. e. $z_1 \in F(E)$, therefore $T(E) \subseteq F(E)$, by $F(E) \subseteq T(E)$ and $T(E) \subseteq F(E)$, we have $T(E) = F(E)$.

2 MAIN RESULTS

Let $S: E \to \mathscr{F}(E)$ is a fuzzy mapping, $O: E \to (0,1]$ is a real-valued function, $\forall\ x \in E$ Let $Gx = (Sx)_{O(x)} = \{u \mid Sx(u) \geqslant O(x)\}$, then $G: E \to 2^E$ is a set-valued mapping, throughout this paper we denote this mapping by G.

THEOREM 2.1 Let (E, F, Δ) is a complete M-PM-spacem, Δ satisfies the condition (1.1), $\Phi: E \to (-\infty, +\infty]$ is a lower semicontinuous function, bounded from below, $\not\equiv +\infty$, $T: E \to E$ is continuous and satisfies the condition (1.2), let $S: E \to \mathscr{F}(E)$ is a fuzzy mapping.

(i) If there exists a real-valued function $O: E \to (0, 1]$ such that $\forall\ x \in E$, $Gx \neq \varnothing$, $TGx = GTx$, and $\exists\ y \in Gx$ such that $\forall\ t > 0, \exists\ t_1 >, t_2 > 0, t_1 + t_2 = t$ satisfies:

$$\Delta(Fy, Tx(t_1), FTx, Ty(t_2)) \geqslant H\left(t - \frac{1}{\alpha}(\Phi(x) - \Phi(y))\right) \qquad (2.1)$$

Where $\alpha > 0$ is a constant, then for every $u \in E$ with $\Phi(Tu) \neq +\infty$ and $\beta > 1$, there exists $p \in E$ such that $p = Tp$ and $Sp(p) \geqslant O(p)$, moreover

$$FTu, p(t) \geqslant H\left(t - \frac{\beta}{\alpha}(T(u) - \Phi(p)\right), \forall\ t > 0 \qquad (2.2)$$

if $\Phi(Tu) \leqslant \inf\limits_{x \in T(E)} \Phi(x) + \varepsilon < \inf\limits_{x \in T(E)} \Phi(x) + 1$, then p satisfies

$$FTu, p(t) \geqslant H\left(t - \frac{\sqrt{\varepsilon}}{2}\right), \forall\ t > 0 \qquad (2.3)$$

(ii) In particular, if $\forall\ x \in E, O(x) = \max\limits_{u \in E} Sx(u)$ satisfies the conditions of (i), then there exists $p \in E$, p is a fuzzy hybrid fixed point of S and T, moreover satisfies (2.2), (2.3).

PROOF. Since $\forall\ x \in E, Gx = (Sx)_{O(x)} = \{u \mid Sx(u) \geqslant O(x)\}, G: E \to 2^E$ is a set-valued mapping, moreover satisfies the conditions of (i), we can difine mapping $Q: E \to E$ such that $\forall\ x \in E, Qx = y$, where $y \in Gx$ and satisfies (2.1), thus $\forall\ x \in E$ which implies that: $\forall\ t > 0, \exists\ t_1 > 0, t_2 > 0, t_1 + t_2 = t$, satisfies:

$$\Delta(FQx, Tx(t_1), FTx, TQx(t_2)) \geqslant H\left(t - \frac{1}{\alpha}(\Phi(x) - \Phi(Qx))\right) \qquad (2.4)$$

by lemman 1.2, $F(E) = T(E)$, moreover $T: E \to E$ is continuous, therefore $F(E)$ is a closed set, thus $(F(E), F, \Delta)$ is a complet M-PM-space. We prove that $\forall\ x \in F(E), Qx \in F(E)$, in fact; $\forall\ x \in F(E)$, since $x = Tx$, $GTx = TGx$, therefore $Qx = QTx \in GTx = TGx \subseteq T(E) = F(E)$, $\therefore Qx \in F(E)$. since $\Phi(Tu) \neq +\infty$, if $\Phi(Tu) = \inf\limits_{x \in T(E)} \Phi(x)$, then $\Phi(QTu) \geqslant \Phi(Tu)$, by $TTu = Tu, \forall\ t > 0, \exists\ t_1 > 0, t_2 > 0, t = t_1 + t_2$ such that

$$\Delta(FQTu, Tu(t_1), FTu, TQTu(t_2)) \geqslant H\left(t - \frac{1}{\alpha}(\Phi(Tu) - \Phi(QTu))\right)$$

we have $\Phi(Tu) = \Phi(QTu)$ ([8]), by $\Phi(Tu) = \Phi(QTu)$, and $TQTu = QTu, \forall\, t >> 0, \exists\, t_1 > 0, t_2 > 0, t = t_1 + t_2$ such that

$$\Delta(F_{QTu,Tu}(t_1), F_{Tu,QTu}(t_2)) \geqslant H(t) \qquad (2.5)$$

we have $F_{QTu,Tu}(t) = H(t), \forall\, t > 0$ ([8]), by $F_{QTu,Tu}(t) = H(t)$, we have $QTu = Tu$, thus taking $p = Tu \in T(E)$, we have $Tp = p = Qp \in Gp$, moreover satisfies (2.2)(2.3).

If $\Phi(Tu) > \inf_{x\in T(E)} \Phi(x)$, then let $\Phi(Tu) - \inf_{x\in T(E)} \Phi(x) = \varepsilon > 0, T(E) = F(E), (T(E), F, \Delta) = (F(E), F, \Delta)$ is a complete M-PM-space, by lemm 1.1 $\forall\, \lambda > 0, \exists\, p \in T(E)$, such that:

$$F_{Tu,p}(t) \geqslant H(t - \frac{1}{\varepsilon\lambda}(\Phi(Tu) - \Phi(p))), \forall\, t > 0 \qquad (2.6)$$

$$F_{Tu,p}(t) \geqslant H(t - \frac{1}{\lambda}), \forall\, t > 0$$

$\forall\, x \in T(E), x \neq p, \exists\, t_0 = t_0(x) > 0$ such that

$$F_{x,p}(t_0) < H(t_0 - \frac{1}{\varepsilon\lambda}(\Phi(p) - \Phi(x))) \qquad (2.7)$$

by (2.7) we have $Qp = p$, in fact, if $Qp \neq p$, then by (2.7), $\exists\, t_0 = t_0(Qp)$ such that $F_{Qp,p}(t_0) < H(t_0 - \frac{1}{\varepsilon\lambda}(\Phi(p) - \Phi(Qp)))$, by (2.4), for $t_0 > 0, \exists\, t_1 > 0, t_2 > 0, t_1 + t_2 = t_0$ such that $\Delta(F_{Qp,Tp}(t_1), F_{Tp,TQp}(t_2)) \geqslant H(t_0 - \frac{1}{\lambda}(\Phi(p) - \Phi(Qp)))$ since $Tp = p, TQp = Qp$, and let $\lambda = \frac{a}{\varepsilon}$, we have

$$\begin{aligned} F_{Qp,p}(t_1) &= F_{Qp,Tp}(t_1) = \Delta(F_{Qp,Tp}(t_1), 1) \\ &\geqslant \Delta(F_{Qp,Tp}(t_1), F_{Tp,TQp}(t_2)) \geqslant H(t_0 - \frac{1}{a}(\Phi(p) - \Phi(Qp))) \\ &= H(t_0 - \frac{1}{\varepsilon\lambda}(\Phi(p) - \Phi(Qp))) > F_{Qp,p}(t_0) \end{aligned}$$

by $t_1 + t_2 = t_0, t_1 < t_0$, this is a contradiction, thus $p = Qp$, it is obvious that $Tp = p = Qp$ and $Qp \in Gp = (Sp)_{O(p)}$, i.e. $Sp(p) \geqslant O(p)$. In particular, if $O(x) = \max_{u\in E} Sx(u)$ satisfies the conditions of (i), it is obvious that p is a fuzzy hybrid fixed point of S and T. By (2.6), $\beta > 1, \Phi(Tu) \geqslant \Phi(p), \lambda = \frac{a}{\varepsilon}$, we have $F_{Tu,p}(t) \geqslant H(t - \frac{1}{\varepsilon\lambda}(\Phi(Tu) - \Phi(p))) \geqslant H(t - \frac{\beta}{a}(\Phi(Tu) - \Phi(p)))$, i.e. (2.2). If $\Phi(Tu) \leqslant \inf_{x\in T(E)} \Phi(x) + \varepsilon < \inf_{x\in T(E)} \Phi(x) + 1$, let $\beta = \frac{1}{\sqrt{\varepsilon}}$, we have $F_{Tu,}$

$p(t) \geqslant H(t-\frac{\beta}{\alpha}(\Phi(Tu)-\Phi(p))) \geqslant H(t-\frac{\beta}{\alpha}\varepsilon) = H(t-\frac{\sqrt{\varepsilon}}{\alpha})$, i. e. (2. 3).

THEOREM 2. 2 Let $(E,F,\Delta),\Phi,T$ satisfies the conditions of theorem 2. 1 moreover $x_1 \neq x_2$ with $Tx_1 \neq Tx_2$, let $S_K: E \to \mathscr{F}(E)(K=1,2,\cdots)$ is a sequence of fuzzy mappings.

(i) If there exists a sequence of functions $O_K: E \to (0,1](K=1,2,\cdots)$ such that $\forall\ x \in E, G_Kx = (S_Kx)_{O_K(x)} \neq \varnothing, TG_Kx = G_KTx$, and $\exists\ y \in G_Kx(K=1,2,\cdots)$ such that $\forall\ t>0, \exists\ t_1>0, t_2>0, t_1+t_2=t$ satisfies:

$$\Delta(Fy,Tx(t_1),FTx,Ty(t_2)) \geqslant H(t-\frac{1}{\alpha}(\Phi(x)-\Phi(y))) \qquad (2.8)$$

where $\alpha>0$ is a constant, then for every $u \in E$ with $\Phi(Tu) \neq +\infty$ and $\beta>1$ there exists $p \in E$ such that $p=Tp, (\bigcap_{K=1}^{+\infty} S_Kp)(p) \geqslant \min_K\{O_K(p)\}$, and (2. 2). (2. 3).

(ii) In particular, if $O_K(x) = \max_{u \in E} S_Kx(u)(K=1,2,\cdots)$ satisfies the conditions of (i), then p is a common fuzzy hybrid fixed point of $\{S_K\}$ and T, moreover satisfies (2. 2), (2. 3).

PROOF. $\forall\ x \in E$, let $Gx = \bigcap_{K=1}^{+\infty} G_Kx$, where $G_Kx = (S_Kx)_{O_K(x)}(K=1,2,\cdots\cdots)$, by assumption $\forall\ x \in E, Gx \neq \varnothing$ and $\exists\ y \in Gx$ such that (2. 8), thus $G: E \to 2^E$ satisfies the condition (2. 1), we shall show that $\forall\ x \in E\ TGx = GTx$.

If $u \in TGx$, then $\exists\ y \in Gx = \bigcap_{K=1}^{+\infty} G_Kx$ with $Ty=u$, thus $y \in G_Kx(K=1,2,\cdots), u=Ty \in TG_Kx = G_KTx(K=1,2,\cdots)$, hence $u \in \bigcap_{K=1}^{+\infty} G_KTx = GTx$, we have $TGx \subseteq GTx$, If $u \in GTx$, i. e. $u \in \bigcap_{K=1}^{+\infty} G_KTx$, then $u \in G_KTx = TG_Kx(K=1,2,\cdots)$, since when $y_1 \neq y_2, Ty_1 \neq Ty_2$, thus $\exists\ y \in G_Kx(K=1,2,\cdots)$ with $Ty=u$, hence $y \in \bigcap_{K=1}^{+\infty} G_Kx = Gx$ with $u=Ty$, thus $u \in TGx$, we have $GTx \subseteq TGx$, we obtain $GTx = TGx$, thus G satifies the conditions of theorem 2. 1, for T and G applying theorem 2. 1, we obtain $p \in E$, such that $p=Tp \in Gp$ and (2. 2)(2. 3), by $p \in Gp = \bigcap_{K=1}^{+\infty} G_Kp = \bigcap_{K=1}^{+\infty} (S_Kp)_{O_K(p)}$, we have $S_Kp(p) \geqslant O_K(p) \geqslant \min_K O_K(p)$, thus $(\bigcap_{K=1}^{+\infty} S_Kp)(p) = \min_K S_Kp(p) \geqslant \min_K O_K(p)$. In particular, if $O_K(x) = \max_{u \in E} S_Kx(u)(K=1,2,\cdots)$, it is easy to deduce that p satisfie $p=Tp$

and $(\bigcap_{K=1}^{+\infty} S_K p)(p) = \max_{u\in E}(\bigcap_{K=1}^{+\infty} S_K p)(u)$, i. e. p is a common fuzzy hybrid fixed point of $\{S_K\}$ and T.

REMARK 2. 1 for a complete metric space (E,d), we can define mapping $F:E \times E \to D$ as follow $Fx,y(t) = H(t-d(x,y)), t \in (-\infty, +\infty)$, it is easy to prove that $(E,F,\min)$ be a complete M—PM—Space, $\Delta = \min$ satisfies the condition (1. 1), therefore it is easy to prove that the theorems of this paper improve and generalize Carist's fixed point theorem and corresponding recent important results of [1—8].

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