FOURIER ANALYSIS ON FUZZY SETS
Part Two

Timur Karaçay

Akdeniz University
Antalya, Turkey
karacay@ufuk.lab.akdeniz.edu.tr

Abstract

Part One of this study was devoted to the construction of the basic mathematical structures on a universe G, and on a set Γ of fuzzy sets in G. They are endowed with structures so that they become locally compact topological groups which are dual to each other. *Part Two* of this study generalizes the concept of proposition and introduces new notions such as logical dependencies, logical bases, logical dimensions. Finally Γ is characterized as a complete orthonormal system in $L^2(G)$ and hence propositions are represented as Fourier series relative to this system.

Prologue

Two-valued logic based on Boolean Algebra has very firm foundations and is the basis of 20th century technology. No mathematician will even doubt the perfection of this system.

However scientists dealing with the mathematical formulations of natural phenomenons realized that the boolean logic lacks something, and consequently, they set forth to open a new page in the history of logics, the so called *fuzzy logics*.

However, this attractive new idea is probably misplaced on the steady, strong foundations of the *boolean logic*.

Thence an innocent question arises: Is the result inevitably a brillant modern sentry-box constructed on an magnificient ancient tower?

If so, scientists may nonetheless feel that it would be worthwile to reconstruct the mathematical basis of fuzzy logics without relying solely on the structure of boolean algebra.

The boolean and fuzzy propositions take their values in the set $\{0, 1\}$ and in the interval $[0, 1]$, in their respective order. We first extend the idea by

assuming that any real or complex valued function defined on our universe G is a proposition.

Secondly, we endow our universe G and the set Γ of fuzzy propositions in G with suitable mathematical structures [2].

Characterization of propositins in $L^2(G)$

It would then be possible to investigate propositions in different spaces of functions. For the sake of simplicity we depart from $L^2(G)$ by imposing the compactness property on the universe G. We shall freely use the notions and terminology of Hilbert spaces.

There remains only to review the core of the theory of Fourier analysis by the replacement the terms *'functions'* with *'propositions'*.

Lemma 1 *If the universe G is compact then its dual Γ is a complete countable orthonormal set of fuzzy propositions in $L^2(G)$.*

The following theorem is the core of the representations of propositions in $L^2(G)$ in terms of the elements of the dual group Γ. The validity is known from the theory of Fourier analysis.

Theorem 1 *Each proposition $f \in L^2(G)$ has a unique Fourier expansion (series) of the form*

$$f(x) = \sum_{\gamma \in \Gamma} \hat{f}(\gamma)(-x, \gamma) \tag{1}$$

where

1. *Each term of the sequence $\{\hat{f}(\gamma) \mid \gamma \in \Gamma\}$ of Fourier coefficients of f is given by $\hat{f}(\gamma) =< f, \gamma >$, $(\gamma \in \Gamma)$.*

2. *The Fourier series converges to f in the norm of $L^2(G)$.*

3. *The sequence $\{\hat{f}(\gamma) \mid \gamma \in \Gamma\}$ belongs to the space l^2 and the equation*

$$||f||_2^2 =< f, f >= \sum_{\gamma \in \Gamma} |\hat{f}(\gamma)|^2 < \infty \tag{2}$$

holds.

4. *If $f, g \in L^2(G)$ then*

$$< f, g >= \sum_{\gamma \in \Gamma} \hat{f}(\gamma)\hat{g}(\gamma) \tag{3}$$

holds.

The last two relations known as Parseval's identity.

As it is known from L^2 theory, the converse of *Theorem 1* is also true. We also state this property for the sake of completeness.

Theorem 2 *A necessary and sufficient condition for a set of complex numbers $c = \{c_\gamma | \ \gamma \in \Gamma\}$ be the Fourier coefficients of some proposition $f \in L^2(G)$ is that $c \in l^2$.*

Logical Dependence, Logical Base, Logical Dimension

Definition 1 *Let A be a any subset of Γ not necesserily finite. Then the set $\Gamma(A)$ of all linear combinations of all finite subsets of A is called the logical span of A.*

$\Gamma(A)$ is the minimal subspace of $L^2(G)$ which contains A.

The followings hold true for all $A, B \subset \Gamma$.

$$\begin{aligned} A \subset B \ &\Rightarrow \ \Gamma(A) \subset \Gamma(B) \\ \Gamma(A \cap B) \ &= \ \Gamma(A) \cap \Gamma(B) \end{aligned}$$

Definition 2 *Two propositions f and g in $L^2(G)$ are said to be logically independent iff $f \perp g$,otherwise, they are said to be logically related to each other.*

Definition 3 *The logical base of $f \in L^2(G)$, denoted by B_f, is defined to be the set*

$$B_f = \cap\{A \mid f \in \Gamma(A)\} \tag{4}$$

Lemma 2 *The logical base of a proposition f is the minimal set A in Γ for which $f \in \Gamma(A)$.*

Lemma 3 *Two propositions f and g are logically independent iff $B_f \cap B_g = \phi$.*

Definition 4 *The logical dimension of a proposion $f \in L^2(G)$ is defined to be the cardinal number of B_f.*

Lemma 4 *$B_\gamma = \{\gamma\}$ for all $\gamma \in \Gamma$, where $\{\gamma\}$ stands for the set whose sole element is γ.*

It follows that $B_\gamma \neq B_\eta$ for all $\gamma, \eta \in \Gamma$ with $\gamma \neq \eta$.

Definition 5 *Two propositions are said to be logically dependent iff they have the same logical base.*

Logical dependency is an equivalence relation in $L^2(G)$. The symbol $[f]$ will denote the equivalence class of f in $L^2(G)$, and $\mathcal{L}^2(G)$ will stand for the set of equivalence classes $[f]$, $f \in L^2(G)$.

Lemma 5 *If f and g are any two propositions in $L^2(G)$ then*

$$f \equiv g \Leftrightarrow [f \in \Gamma(A) \Leftrightarrow g \in \Gamma(A),\ (A \subset \Gamma)].$$

Definition 6 *We define the binary operations $\wedge$ and $\vee$ on $\mathcal{L}^2(G)$ by the relations*

$$[f] \wedge [g] = [f] \cap [g]$$

and

$$[f] \vee [g] = [f] \cup [g].$$

$[f] \wedge [g]$ is the equivalence class whose base is $B_f \cap B_g$.

Definition 7 *In particular, we define the operation $f \wedge g$ to be any element in $[f] \wedge [g]$.*

Definition 8 *Two equivalence classes $[f]$ and $[g]$ are said to be logically independent on G iff $[f] \wedge [g] = [0]$, or equivalently, $B_f \cap B_g = \phi$.*

Lemma 6 *Two propositions f and g in $L^2(G)$ are logically independent on G iff the classes $[f]$ and $[g]$ are logically independent.*

Lemma 7 *$[\gamma] = \{\gamma\}$ for all $\gamma \in \Gamma$, where $\{\gamma\}$ stands for the set whose sole element is γ.*

Lemma 8 *Different characters are logically independent.*

Note that, if p is a boolean or fuzzy proposition, then $[p] = \{p\}$ which is a constant value in the interval $[0, 1]$. Therefore, we have

$$[p] \wedge [q] = \{p\} \cap \{q\}.$$

for any two boolean or fuzzy propositions p and q. Hence the sole element of $[p] \wedge [q]$ is $p \wedge q$. This simply means that, the operation $\wedge$ defined on the set of equivalence classes $\mathcal{L}^2(G)$ is a generalization of the operation $\wedge$ defined on the boolean or fuzzy propositions.

Similar result can also be stated for the operation $\vee$.

The Implication

Definition 9 *We say that the class $[f]$ implies the class $[g]$, written $[f] \Rightarrow [g]$, iff $B_f \subset B_g$. Similarly, the proposition f is said to imply g iff $[f] \Rightarrow [g]$.*

Lemma 9 *We have the implication $[0] \Rightarrow [f]$ for any proposition f.*

Going to individual propositions, we obtain $0 \Rightarrow f$ for any proposition f. In particular, we have $0 \Rightarrow 0$, $0 \Rightarrow 1$ and $1 \Rightarrow 1$ proving the important assumptions in boolean logic.

Epilogue

The characterization of propositions presented in this work uses nothing but the powerfull mathematical tools men invented in the past of mathematics.

REFERENCES

1. Hewitt,E-Ross,K.A. *Abstract Harmonic Analysis I-II.* Springer-Verlag, Berlin, 1970.

2. Karacay,T., *Fourier Analysis on Fuzzy Sets (Part One).* BUSEFAL, April, 1997.

3. Loomis,L.H. *An Introduction to Abstract Harmonic Analysis.* Van Nostrand, New York,1953.

4. Pryce,J.D. *Basic Methods of Linear Functional Analysis.* Hutchinson & Co. Ltd., London, 1973.

5. Rudin,W. *Fourier Analysis on Groups.* Interscience,New York,1962.

6. Zygmund,A. *Trigonometrical Series,* Monografje Matematyczne, Warsaw-Lwów, 1935.