The Connectedness and Local Connectedness in Induced $I(L)$-Fuzzy Topological Spaces*

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Abstract: The main purpose of this paper is to prove that the induced $I(L)$-fuzzy topological space preserves the connectedness and the local connectedness.

Keywords: Induced $I(L)$-fuzzy topological spaces, connectedness, local connectedness.

1. Introduction

In [4], Wang introduced the concept of induced $I(L)$-fuzzy topological spaces by using the $I(L)$-valued lower semicontinuous mappings (Kubiak [2]) and proved that this kind of induced space preserves the Cartesian product and the N-compactness. In this paper, we continue with investigation of induced $I(L)$- fuzzy topological spaces. we prove that the L-fuzzy topological space (L^X, δ) is connected (locally connected) if and only if the induced $I(L)$-fuzzy topological space $(I(L)^X, \omega(\delta))$ is connected (locally connected).

2. Induced $I(L)$-fuzzy topological spaces and some lemmas

Throughout this paper L denotes a fuzzy lattice, i.e., a completely distributive lattice with an order-reversing involution $\alpha \to \alpha'$, 0 and 1 are its smallest and greatest elements, respectively. Given a nonempty set X, (L^X, δ) denotes an L-fuzzy topological space, briefly L-fts. Let $I = [0, 1]$, and $I(L)$ denotes the L-fuzzy

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unit interval [1]. A partial order on $I(L)$ is naturally defined by $[\lambda] \leq [\mu]$ iff $\lambda(t-) \leq \mu(t-)$ and $\lambda(t+) \leq \mu(t+)$ for all $t \in I$. For any $[\lambda]$, $[\mu] \in I(L)$, define $[\lambda] \vee [\mu] = [\lambda \vee \mu]$ and $[\lambda] \wedge [\mu] = [\lambda \wedge \mu]$. Moreover, let $\overline{\lambda} : R \to L$ satisfying $\overline{\lambda}(t) = \lambda(1-t)'$ for all $t \in R$ and define $[\lambda]' = [\overline{\lambda}]$. To simplify notation, we shall identify equivalence classes $[\lambda]$, $[\mu]$ with their representatives in the sequel. By [4] we know that $(I(L), \leq, \vee, \wedge, ')$ is a fuzzy lattice, and λ is an irreducible element in $I(L)$ iff there exist an irreducible element $\alpha \in L$ and $t \in I$ such that $\lambda = \lambda_{\alpha,t}$, where $\lambda_{\alpha,t} \in I(L)$ is defined as follows:

$$\lambda_{\alpha,\,t}(s+) = \begin{cases} 1, & s < 0, \\ \alpha, & 0 \leq s < t, \\ 0, & t \leq s. \end{cases}$$

Definition 2.1 (Wang [4]). Let $t \in I$. Define mappings σ_t, $\omega_t : I(L)^X \to L^X$ satisfying $\sigma_t(\mu) = \mu^{-1}(R_t) = R_t \circ \mu, \omega_t(\mu) = \mu^{-1}(L_t') = L_t' \circ \mu$ for each $\mu \in I(L)^X$, where $L_t[\lambda] = \lambda(t-)'$, $R_t[\lambda] = \lambda(t+)$.

Definition 2.2 (Kubiak [2]). Let $(L^X. \delta)$ be an L-fts. A mapping $\mu : X \to I(L)$ is called $I(L)$-valued lower semicontinuous if $\sigma_t(\mu) \in \delta$ for each $t \in I$.

Definition 2.3 (Wang [4]). Let $(L^X. \delta)$ be an L-fts. The set of all $I(L)$-valued lower semicontinuous mappings on X, being an $I(L)$-fuzzy topology, is called an induced $I(L)$-fuzzy topology which is denoted by $\omega(\delta)$. $(I(L)^X, \omega(\delta))$ is called an induced $I(L)$-fuzzy topological space, or simply induced $I(L)$-fts.

Definition 2.4 (Wang [4]). Define the mapping $^* : L^X \to I(L)^X$ satisfying

$$A^*(x)(t+) = \begin{cases} 1, & t < 0, \\ A(x), & 0 \leq t < 1, \\ 0, & t \geq 1, \end{cases}$$

for each $A \in L^X$ and each $x \in X$. Moreover, for each $t \in \mathbf{R}$, define a constant mapping $t^* : X \to I(L)$ by letting

$$t^*(x)(s+) = \begin{cases} 1, & s < t, \\ 0, & s \geq t. \end{cases} \quad \text{for each } x \in X.$$

Obviously, 0^* is the smallest element in $I(L)^X$. It is easy to prove that the operators $\omega_t, \sigma_t, \omega$ and * have the following properties:

Proposition 2.1 (1) *ω_t preserves finite sups and arbitrary infs;*
(2) *σ_t preserves arbitrary sups and finite infs;*
(3) ** preserves arbitrary sups and arbitrary infs.*

Proposition 2.2 *Let $\mu \in I(L)^X$, $A \in L^X$. Then*
(1) *$\mu \in \omega(\delta)'$ iff $\omega_t(\mu) \in \delta'$ for each $t \in I$;*

(2) $A \in \delta'$ iff $A^* \in \omega(\delta)'$.

Proposition 2.3 *Let $\mu \in I(L)^X$, $A \in L^X$ and $t \in I$. Then the following equalities hold:*

(1) $(\sigma_t(\mu))' = \omega_{1-t}(\mu')$, $\quad (\omega_t(\mu))' = \sigma_{1-t}(\mu')$;

(2) $(A^*)' = (A')^*$;

(3) $(t^*)' = (1-t)^*$ *for all* $t \in I$;

(4) $\sigma_t(A^*) = A$ *for* $t \neq 1$, $\quad \omega(A^*) = A$.

To verify our main results, we need several lemmas:

Lemma 2.1 *Let λ, $\mu \in I(L)$. Then the following statements are equivalent:*

(1) $\lambda = \mu$ $(\lambda \leq \mu)$;

(2) $\lambda(t+) = \mu(t+)$ $(\lambda(t+) \leq \mu(t+))$ *for all* $t \in I$;

(3) $\lambda(t-) = \mu(t-)$ $(\lambda(t-) \leq \mu(t-))$ *for all* $t \in I$.

Proof. Only note that $\lambda(t-) = \bigwedge\{\lambda(s+) \mid s < t\}$, $\lambda(t+) = \bigvee\{\lambda(s-) \mid s > t\}$ for all $\lambda \in I(L)$ and all $t \in \mathbf{R}$.

Lemma 2.2 *Let $\eta(x_\alpha)$ denotes the set of all closed R-neighborhoods of a molecule x_α in (L^X, δ), and $\eta(x_{\lambda_{\alpha,t}})$ denotes the set of all closed R-neighborhoods of a molecule $x_{\lambda_{\alpha,t}}$ in $(I(L)^X, \omega(\delta))$.*

(1) *If $P \in \eta(x_\alpha)$, then $P^* \vee s^* \in \eta(x_{\lambda_{\alpha,t}})$ for all $t \in (0,1]$ and all $s \in [0,t)$.*

(2) *If $P \in \eta(x_\alpha)$, then $P^* \in \eta(x_{\lambda_{\alpha,t}})$ for all $t \in (0,1]$.*

(3) *If $P \in \eta(x_{\lambda_{\alpha,t}})$, then there exists an $s \in (0,t)$ such that $\omega_s(P) \in \eta(x_\alpha)$.*

Proof. (1) Let $P \in \eta(x_\alpha)$. Then for any $t \in (0,1]$ and $s \in [0,t)$ we have

$$\lambda_{\alpha,t}(s+) = \alpha \not\leq P(x) = P^*(x)(s+) = (P^* \vee s^*)(x)(s+),$$

and so $\lambda_{\alpha,t} \not\leq (P^* \vee s^*)(x)$, i.e., $x_{\lambda_{\alpha,t}} \not\leq P^* \vee s^*$. Since $P \in \delta'$, $(P')^* \wedge (1-s)^* \in \omega(\delta)$ by [4, Lemma 3.1]. This implies $P^* \vee s^* \in \omega(\delta)'$. Therefore $P^* \vee s^* \in \eta(x_{\lambda_{\alpha,t}})$.

(2) Immediate from (1).

(3) Let $P \in \eta(x_{\lambda_{\alpha,t}})$. Then $\lambda_{\alpha,t} \not\leq P(x)$, and so there exists an $s_0 \in [0,t)$ such that $\alpha = \lambda_{\alpha,t}(s_0+) \not\leq P(x)(s_0+)$. Taking $s \in (s_0, t)$, we have $P(x)(s-) \leq P(s_0+)$. Hence $\alpha \not\leq P(x)(s-) = \omega_s(P)(x)$. Since $P \in \omega(\delta)'$, by Proposition 2.2 $\omega_s(P) \in \delta'$. Therefore $\omega_s(P) \in \eta(x_\alpha)$.

Lemma 2.3 *The mapping $\sigma_0 : (I(L)^X, \omega(\delta)) \to (L^X, \delta)$ is a continuous order-homomorphism.*

3. Main Results

Definition 3.1 (Wang [5]). Let (L^X, δ) be an L-fts, and A, $B \in L^X$. A and B is said to be disjoint if $A^- \wedge B = A \wedge B^- = 0$.

Definition 3.2 (Wang [5]). Let (L^X, δ) be an L-fts, and $A \in L^X$. A is called a connected set if it is not the union of two disjoint nonzero L-fuzzy sets. In particular, if $1 \in L^X$ is a connected set, then (L^X, δ) is called a connected L-fts.

Definition 3.2 (Wang and Shi [6]). L-fts (L^X, δ) is called locally connected, if for each $x_\alpha \in M^*(L^X)$ and $P \in \eta(x_\alpha)$ there exists $Q \in \eta(x_\alpha)$ such that $P \leq Q$ and Q' is connected.

Theorem 3.1. *Let (L^X, δ) be an L-fts, and $A \in L^X$. Then A is connected in (L^X, δ) iff A^* is connected in $(I(L)^X, \omega(\delta))$.*

Proof. *Necessity.* Assume that A^* is not connected. Then there exist two nonzero elements $B,\ C \in I(L)^X$, such that $A^* = B \vee C$ and $B^- \wedge C = B \wedge C^- = 0$. We choose $x,\ y \in X$ and $r,\ s \in (0,1]$ such that $B(x)(r-) \neq 0,\ C(y)(s-) \neq 0$. Taking $t = \min\{r, s\}$, then $A = \omega_t(A^*) = \omega_t(B) \vee \omega_t(C)$, where $\omega_t(B)$ and $\omega_t(C)$ are nonzero. By Proposition 2.2, $\omega_t(B^-) \in \delta'$. Hence, by Proposition 2.1 we have

$$\omega_t(B)^- \wedge \omega_t(C) \leq \omega_t(B^-) \wedge \omega_t(C) = \omega_t(B^- \wedge C) = 0.$$

Similarly, we can prove $\omega_t(B) \wedge \omega_t(C)^- = 0$. Therefore A is not connected.

Sufficiency. Assume that A is not connected. Then there exist two nonzero elements $B,\ C \in L^X$, such that $A = B \vee C$ and $B^- \wedge C = B \wedge C^- = 0$. Obviously, $B^*,\ C^*$ are also nonzero, and $A^* = B^* \vee C^*$. By Proposition 2.2 and 2.1, it is easy to prove that $(B^*)^- \wedge C^* = B^* \wedge (C^*)^- = 0^*$. Therefore A^* is not connected.

Corollary 3.1. *(L^X, δ) is connected iff $(I(L)^X, \omega(\delta))$ is connected.*

Theorem 3.2. *Let (L^X, δ) be an L-fts, and A be a connected L-fuzzy set in (L^X, δ). Then for each $t \in (0,1]$, $A^* \wedge t^*$ is a connected $I(L)$-fuzzy set in $(I(L)^X, \omega(\delta))$.*

Proof. Analogous to necessity of Theorem 3.1 and note that $\omega_s(A^* \wedge t^*) = A$ for all $s \in (0, t]$.

Theorem 3.3. *Let A be a connected $I(L)$-fuzzy set in $(I(L)^X, \omega(\delta))$. Then $\sigma_0(A)$ is a connected L-fuzzy set.*

Proof. Assume that $\sigma_0(A)$ is not connected. Then there exist nonzero $B_0,\ C_0 \in L^X$ such that $\sigma_0(A) = B_0 \vee C_0$, and $B_0^- \wedge C_0 = B_0 \wedge C_0^- = 0$. Define $B,\ C \in I(L)^X$ as follows:

$$B(x)(t+) = \sigma_t(B)(x), \quad C(x)(t+) = \sigma_t(C)(x)$$

for all $x \in X$ and all $t \in [0,1)$. By Lemma 2.1, it is easy to know that $A = B \vee C$. Since σ_0 is a continuous order-homomorphism (Lemma 2.3), we have $\sigma_0(B^-) \leq \overline{\sigma_0(B)}$ and $\sigma_0(C^-) \leq \overline{\sigma_0(C)}$. Thus we can prove that $B^- \wedge C = B \wedge C^- = 0^*$. This shows that A is not connected.

Theorem 3.4. (L^X, δ) is locally connected iff $(I(L)^X, \omega(\delta))$ is locally connected.

Proof. *Necessity.* Let (L^X, δ) is locally connected and $x_{\lambda_\alpha, t} \in M^*(I(L)^X)$. For any $P \in \eta(x_{\lambda_\alpha, t})$, there exists an $s \in (0, t)$ such that $\omega_s(P) \in \eta(x_\alpha)$ by Lemma 2.2. Since (L^X, δ) is locally connected, there exists $Q \in \eta(x_\alpha)$ such that $\omega_s(P) \leq Q$, and Q' is connected. It is easy to see that $P \leq Q^* \vee s^*$ and $Q^* \vee s^* \in \eta(x_{\lambda_\alpha, t})$. By Proposition 2.3, $(Q^* \vee s^*)' = (Q')^* \wedge (1-s)^*$. Note that Q' is connected, from Theorem 3.2 we know that $(Q^* \vee s^*)'$ is connected in $(I(L)^X, \omega(\delta))$. Hence $(I(L)^X, \omega(\delta))$ is locally connected.

Sufficiency. Let $(I(L)^X, \omega(\delta))$ is locally connected and $x_\alpha \in M^*(L^X)$. For any $P \in \eta(x_\alpha)$, we know that $P^* \in \eta(x_{\lambda_\alpha, 1})$ by Lemma 2.2. Since $(I(L)^X, \omega(\delta))$ is locally connected, there exists $Q \in \eta(x_{\lambda_\alpha, 1})$ such that $P^* \leq Q$ and Q' is connected. For $Q \in \eta(x_{\lambda_\alpha, 1})$, by Lemma 2.2 there exists $s \in (0, 1)$ such that $\omega_s(Q) \in \eta(x_\alpha)$. Notice that $P^* \leq Q$ implies $P \leq \omega_t(Q)$ for all $t \in (0, 1]$. Hence $\bigwedge_{t \in (0,1]} \omega_t(Q) \in \eta(x_\alpha)$. By Proposition 2.3 and Theorem 3.3, we know that

$$(\bigwedge_{t \in (0,1]} \omega_t(Q))' = \bigvee_{t \in (0,1]} \sigma_{1-t}(Q') = \sigma_0(Q')$$

is connected in (L^X, δ). Therefore (L^X, δ) is locally connected.

Corollary 3.2 The connectedness (local connectedness) is a good extension in the sence of Lowen [3].

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