On the weak convergence of sequences of fuzzy measures

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Abstract In this paper, in the sugeno's fuzzy measures space, we first put forward the concepts of the weak convergence and the metric of fuzzy measures. And then, a few equivalent conditions on the weak convergence of sequences of fuzzy measures are given. Finally, in the sense of this metric, we obtain that the family of fuzzy measures constitute a separable metric space.

Keywords Fuzzy measure; fuzzy integral; weak convergence; metric; density.

1 Introduction

Since sugeno [6] proposed the concepts of fuzzy measures and fuzzy integrals. The theory has been made deeper and wider by Ralescu and Adams [2], Wang [3], Zhang [4] and many others. In particularly, on the continuity of the fuzzy integral with respect to different kinds of convergence has been exhaustively studied in the last years.

But up to now, there are only a few discussions on the convergence of sequences of fuzzy measures. Ralescu and Adams [2] proved theorems of continuity of the fuzzy integral with respect to measure convergence and pointwise convergence. Wang [3] used the concepts of autocontinuity and null - additivity for a continuous fuzzy measure, improving the Ralescu and Adams' results.

The main purpose of this paper is to define the weak convergence by means of sugeno's fuzzy integrals, discuss the metric of fuzzy measures and the equivalent conditions on the weak convergence of sequences of fuzzy measures.

The structure of this paper is as follows. As a preparation in section 2, we first remind some concepts and results that will be used in the article. In section 3, we introduce the concepts on the weak convergence of sequences of fuzzy measures and the metric of fuzzy measures. In the meantime, the equivalent conditions on the weak convergence and the uniqueness of the limit are being studied. In section 4, making use of the concept of the metric of fuzzy measures, we first obtain a necessary and sufficient conditions on the weak convergence of sequences of fuzzy measures. Consequently, we prove that the family of fuzzy measures constitute a separable metric space.

2 Preliminaries

In this section, we will introduce some notations which will be used in the paper, and meanwhile we will remind some concepts and results on fuzzy measures and fuzzy integrals as preparation. Next we give a important conclusion (proposition 2.3) in general topology.

Let X be a distance space, $\mathscr{A}$ denote the σ - algebra formed by the family of all open sets in X. We call the element in $\mathscr{A}$ a Borel set in X. $(X, \mathscr{A})$ is said to be the Borel measurable space, Obviously, $\mathscr{A}$ is also the σ - algebra formed by the family of all closed sets in X. $P_O(X)$ and $P_C(X)$ denote the family of all open sets and all closed sets on X respectively, $C_+(X)$ be the set of all non - negative bounded continuous functions on X.

Definition 2.1 A set - function $\mu : \mathscr{A} \to [0, +\infty]$ is said to be a fuzzy measure if it satisfies the following conditions:

(1) $\mu(\phi) = 0$;

(2) If $A, B \in \mathscr{A}$ and $A \subset B$ implies $\mu(A) \leqslant \mu(B)$;

(3) If $\{A_n\}_{n=1}^{\infty} \subset \mathscr{A}$ and $A_n \uparrow$ implies $\lim_{n\to\infty} \mu(A_n) = \mu(\lim_{n\to\infty} A_n)$

(4) If $\{A_n\}_{n=1}^{\infty} \subset \mathscr{A}$, $A_n \downarrow$ and there exists a $n_0 \in N$ such that $\mu(A_{n_0}) < +\infty$ implies $\lim_{n\to\infty} \mu(A_n) = \mu(\lim_{n\to\infty} A_n)$.

The triplet $(X, \mathscr{A}, \mu)$ is called a fuzzy measure space. Let $FM(X)$ denote the set of all fuzzy measures on $(X, \mathscr{A})$

Definition 2.2 Let $\mu, \upsilon \in FM(X)$, if for an arbitrary closed set $A \in P_C(X)$ (or an arbitrary open set $A \in P_O(X)$), $\mu(A) = \upsilon(A)$ always holds, then we say that μ equals υ, and denote as $\mu = \upsilon$.

Definition 2.3 Let $(X, \mathscr{A}, \mu)$ be a fuzzy measure space, f be a non - negative measurable function on $(X, \mathscr{A})$, $A \in \mathscr{A}$. Then the fuzzy integral of f on A with respect to μ is defined as

$$\int_A f d\mu = \bigvee_{\alpha>0} [\alpha \wedge \mu(A \cap \{f \geqslant \alpha\})]$$

The following some interesting properties of the fuzzy integral are well known.

Proposition 2.1 [2] (Transformation theorem of fuzzy integrals) Let f be a non - negative measurable function on $(X, \mathscr{A})$, $A \in \mathscr{A}$, $\mu \in FM(X)$. Then

$$\int_A f d\mu = \int_0^{+\infty} \mu(A \cap \{f \geqslant \alpha\}) d\alpha$$

Where α is the Lebesgue measure on $[0, +\infty]$, and the right - hand side integral $\int_0^{+\infty} \mu(A \cap \{f \geqslant \alpha\}) d\alpha$ is also a fuzzy integral.

Applying transformation theorem of fuzzy integrals, we are easy to obtain the following conclusions.

Proposition 2.2 (Ralescu and Adams [2]). Let $(X, \mathscr{A}, \mu)$ be a fuzzy measure space, $A \in \mathscr{A}$, Then

(1) $\int_X \chi_A d\mu = \mu(A)$, where χ_A *is* a characteristic function on A;

(2) If f and g are all non - negative measurable functions on $(X, \mathscr{A})$, and $f(x) \leqslant g(x)$, *for each* $x \in A$. Then $\int_A f d\mu \leqslant \int_A g d\mu$;

(3) Let f and g be non-negative bounded measurable functions, defining $\| f - g \| = \sup_{x \in A} | f(x) - g(x) |$, then for any $\varepsilon > 0$, whenever $\| f - g \| < \varepsilon$, we have $| \int_A f d\mu - \int_A g d\mu | < \varepsilon$.

According to the requirement in the sequel, we give an important result in the theory of the classical sets as follows:

Proposition 2.3. [8]. Let (X, d) be a metric space, F and G be two pairwise disjoint closed sets, i.e., $F \cap G = \phi$. Then there exists a real valued continuous function f such that the following conditions are fulfilled.

(1) $0 \leqslant f(x) \leqslant 1$ for arbitrary $x \in X$.

(2) $f(x) = \begin{cases} 1 & x \in F \\ 0 & x \in G \end{cases}$.

Furthermore, if we have $\inf \{ d(x, y) \mid x \in F, y \in G \} = \delta > 0$ besides above, then the function f is uniformly continuous on X.

Note: We may let the function $f(x) = \dfrac{d(x, G)}{d(x, F) + d(x, G)}$, where $x \in X$, $d(x, F) + d(x, G) > 0$.

3 Weak convergence and metric.

In this section, on the family of fuzzy measures, we first put forward the concept of the weak convergence, and then study the equivalent conditions of the weak convergence of sequences of fuzzy measures and the uniqueness of its limit. Finally, we introduce the definition of the metric of fuzzy measures, and express its validity.

From now on, we always assume that X is a separable distance space.

Definition 3.1 Let X be a separable distance space, $A \in \mathscr{A}$, $\{ \mu_n, \mu \} \subset FM(X)$, $n = 1, 2, \cdots$. If $\lim_{n \to \infty} \int_A f d\mu_n = \int_A f d\mu$, for every $f \in C_+(X)$, then we say that the sequence of fuzzy measures $\{ \mu_n \}$ weak converges to μ. write $\mu_n \overset{w}{\Rightarrow} \mu$.

From this definition, we are easy to derive from the following theorem.

Theorem 3.1 Let (X, d) be a separable distance space, $A \in \mathscr{A}$, $\{ \mu_n, \mu \} \subset FM(X)$, $n = 1, 2, \cdots$. Then the following conditions are equivalent:

(1) $\mu_n \overset{w}{\Rightarrow} \mu$;

(2) $\lim_{n \to \infty} \mu_n(A) = \mu(A)$ and $\overline{\lim}_{n \to \infty} \mu_n(F) \leqslant \mu(F)$, for any closed set $F \in P_C(X)$;

(3) $\lim_{n \to \infty} \mu_n(A) = \mu(A)$ and $\underline{\lim}_{n \to \infty} \mu_n(U) \geqslant \mu(U)$, for any open set $U \in P_O(X)$.

Proof. (1)⇒(2) , Let $f = \chi_A$, by proposition 2.2 (1), we have $\int_X \chi_A d\mu = \mu(A)$, $\int_X \chi_A d\mu_n = \mu_n(A)$. Hence $\lim_{n\to\infty}\mu_n(A) = \mu(A)$.

In addition, for arbitrary closed set $F \in P_C(X)$, we constitute a set

$G_k = \{x \in X \mid d(x,F) < \frac{1}{k}\}, k = 1,2,\cdots$.

Obviously, we have $F = \bigcap_{k=1}^{\infty} G_k$ and $F \cap G'_k = \phi$. Where G'_k is a complement set of G_k and $G'_k \in P_C(X)$.

In the meantime, $\inf\{d(x,y) \mid x \in F, y \in G'_k\} \geqslant \frac{1}{k} > 0$

Take advantage of proposition 2.3, for each natural number k , there exists an uniformly continuous function $f_k(x)$ such that $0 \leqslant f_k(x) \leqslant 1$, for arbitrary $x \in X$, and $f_k(x) = \begin{cases} 1 & x \in F \\ 0 & x \in G'_k \end{cases}$, consequently, $\chi_F \leqslant f_k \leqslant \chi_{G_k}, k = 1,2,\cdots$.

Hence, we assert that $\overline{\lim_{n\to\infty}}\mu_n(F) = \overline{\lim_{n\to\infty}}\int_X \chi_F d\mu_n \leqslant \overline{\lim_{n\to\infty}}\int_X f_k d\mu_n$

$$= \int_X f_k d\mu \leqslant \int_X \chi_{G_k} d\mu = \mu(G_k).$$

Let $k \to \infty$, since $G_k \downarrow F$. Take limit on the two hands of above,

we get that $\overline{\lim_{n\to\infty}}\mu_n(F) \leqslant \lim_{k\to\infty}\mu(G_n) = \mu(\lim_{k\to\infty} G_k) = \mu(\lim_{k\to\infty} G_k) = \mu(F)$.

(2)⇒(3) . Simulate above proof, it is obvious.

(3)⇒(1) We may prove it by the methods of constituting the sets and the functions. It is complex, so we omit it.

Now, we will discuss the weak convergence of sequences of fuzzy measures and the uniqueness of its limit. Let $U_+(X)$ be the set of all uniformly continuous function on X .

Theorem 3.2 Let (X,d) be a separable distance space, $\mu, \upsilon \in FM(X), A \in \mathscr{A}$ is a fixed set. If $\int_A f d\mu = \int_A f d\upsilon$, for all $f \in U_+(X)$, then $\mu = \upsilon$.

Proof. For arbitrary closed set $F \in P_C(X)$, we constitute the set

$G_n = \{x \in X \mid d(x,F) < \frac{1}{n}\}, n = 1,2,\cdots$.

In the light of proposition 2.3, there exists an uniformly continuous function f_n such that $0 \leqslant f_n \leqslant 1$ *and* $f_n(x) = \begin{cases} 1 & x \in F \\ 0 & x \in G'_n \end{cases}$

Thus we obtain $0 \leqslant \chi_F \leqslant f_n \leqslant \chi_{G_n}$.

By(1) and (2) in proposition 2.2, it follows that

$\mu(F) = \int_X \chi_F d\mu \leqslant \int_X f_n d\mu = \int_X f_n d\upsilon \leqslant \int_X \chi_{G_n} d\upsilon = \upsilon(G_n)$

Let $n \to \infty$, because of $G_n \downarrow F$, hence we have

$\mu(F) \leqslant \lim_{n\to\infty} \upsilon(G_n) = \upsilon(\lim_{n\to\infty} G_n) = \upsilon(F)$.

In the course of the above proof, Commuting the position of μ and υ, then we get immediately that $\mu(F) \geqslant \upsilon(F)$.

Consequently, $\mu(F) = \upsilon(F)$, For each $F \in P_C(X)$, from definition 2.2, we have $\mu = \upsilon$. And so the uniqueness is proved.

Definition 3.2 Let (X, d) be a separable distance space, $CD(X) = \{f_i \mid f_i \in C_+(x)$, $i = 1, 2, \cdots$ be a countable dense subset of $C_+(X)$, $A \in \mathscr{A}$. Let $f_1 = \chi_A$, for arbitrary $\mu_1, \mu_2 \in FM(X)$, we define the function

$$\rho : FM(X) \times FM(X) \to [0, 1]$$

$$\rho(\mu_1, \mu_2) = \sum_{i=1}^{\infty} \frac{1}{2^i} \min\{1, \mid \int_A f_i d\mu_1 - \int_A f_i d\mu_2 \mid\},$$

where $f_i \in CD(X)$, $i = 1, 2, \cdots$.

Utilizing this definition. we are easy to get the following results.

Theorem 3.3 $(FM(X), \rho)$ constitutes a metric space.

Proof. First, in the light of definition 3.2, for any $\mu_1, \mu_2 \in FM(X)$, we know easily that $\rho(\mu_1, \mu_2) \geqslant 0$ *and if* $\mu_1 = \mu_2$, then we derive from $\rho(\mu_1, \mu_2) = 0$

Furthermore, $\rho(\mu_1, \mu_2) \leqslant \rho(\mu_1, \mu_3) + \rho(\mu_3, \mu_2)$ is easy to be verified. where $\mu_3 \in FM(X)$.

Second, if $\rho(\mu_1, \mu_2) = 0$, then we have $\int_A f_i d\mu_1 = \int_A f_i d\mu_2$, for all $f_i \in CD(X)$, $i = 1, 2, \cdots$. Since the set $CD(X)$ is dense in $C_+(X)$, it follows that $\int_A f d\mu_1 = \int_A f d\mu_2$, for all $f \in C_+(X)$

Take advantage of theorem 3.2, we conclude that $\mu_1 = \mu_2$.

Thus, $(FM(X), \rho)$ is a metric space.

4. Conditions on the weak convergence of sequences of fuzzy measures.

In this section, on the basis of the definition of ρ given before, we first study the necessary and sufficient conditions on the weak convergence of sequences of fuzzy measures. Second, we prove that the family of fuzzy measures constitute a separable metric space.

Theorem 4.1 Let $(FM(X), \rho)$ be a metric space, $\{\mu_n, \mu\} \subset FM(X)$, $n = 1, 2, \cdots$, $A \in \mathscr{A}$, CD(X) be a countable dense subset of $C_+(X)$. Then $\mu_n \overset{w}{\Rightarrow} \mu$ iff $\rho(\mu_n, \mu) \to 0$.

Proof Necessity, since $\mu_n \overset{w}{\Rightarrow} \mu$, it shows that $\lim\limits_{n\to\infty}\int_A f_i d\mu_n = \int_A f_i d\mu$, $f_i \in CD(X)$, $i = 1, 2, \cdots$, Then for arbitrary $\varepsilon > 0$, we may always choose a nutural number m_0 such that $\frac{1}{2^{m_0}} < \varepsilon$. Thus we assert that

$$\rho(\mu_n, \mu) \leqslant \sum_{i=1}^{m_0} \frac{1}{2^i} \min\{1, \left|\int_A f_i d\mu_n - \int_A f_i d\mu\right|\} + \sum_{i=m_0+1}^{\infty} \frac{1}{2^i}$$

$$< \sum_{i=1}^{m_0} \frac{1}{2^i} \min\{1, \left|\int_A f_i d\mu_n - \int_A f_i d\mu\right|\} + \varepsilon$$

Let $n \to \infty$, then we have $\overline{\lim\limits_{n\to\infty}} \rho(\mu_n, \mu) \leqslant \varepsilon$.

Letting $\varepsilon \to 0$ implies $\lim\limits_{n\to\infty} \rho(\mu_n, \mu) = 0$ or $\rho(\mu_n, \mu) \to 0, (n \to \infty)$

Sufficient. If $\rho(\mu_n, \mu) \to 0, (n \to \infty)$, by definition 3.2,

It follows that $\lim\limits_{n\to\infty}\int_A f_i d\mu_n = \int_A f_i d\mu$, where $f_i \in CD(X), i = 1, 2, \cdots$.

It means that for arbitrary $\varepsilon > 0$, there exists a natural number N, whenever $n > N$, for all $f_i \in CD(X)$,

$$\left|\int_A f_i d\mu_n - \int_A f_i d\mu\right| < \frac{\varepsilon}{3} \quad \cdots\cdots \quad (1)$$

Thus, for every $f \in C_+(X)$, whenever $\| f - f_i \| < \varepsilon$ (the norm $\| \cdot \|$ always exists on the space $C_+(X)$), from (3) in proposition 2.2, we obtain that)

$$\left|\int_A f d\mu_n - \int_A f_i d\mu_n\right| < \frac{\varepsilon}{3} \quad \cdots\cdots \quad (2)$$

$$\left|\int_A f_i d\mu - \int_A f d\mu\right| < \frac{\varepsilon}{3} \quad \cdots\cdots \quad (3)$$

Consequently, $\left|\int_A f d\mu_n - \int_A f d\mu\right| \leqslant \left|\int_A f d\mu_n - \int_A f_i d\mu_n\right|$

$$+ \left|\int_A f_i d\mu_n - \int_A f_i d\mu\right| + \left|\int_A f_i d\mu - \int_A f d\mu\right|$$

where $f_i \in CD(X), i = 1, 2, \cdots$.

By density of the set $CD(X)$, combining with (1), (2), (3),

we get that $\left|\int_A f d\mu_n - \int_A f d\mu\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$.

i.e., $\lim\limits_{n\to\infty}\int_A f d\mu_n = \int_A f d\mu$.

Therefore, $\mu_n \overset{w}{\Rightarrow} \mu$, the theorem is completed.

Theorem 4.2 $(FM(X), \rho)$ constitutes a separable metric space

Proof We only need prove that $(FM(X), \rho)$ is separable.

Let $A \in \mathscr{A}$, First, we consider the mapping H from $FM(X)$ to R^∞, where $R^\infty = R \times R \times \cdots \times R \times \cdots$.

For arbitrary $\mu \in FM(X), A \in \mathscr{A}$, we define the mapping

$$H: FM(X) \to R^\infty$$
$$\mu \to H(\mu) = \left(\int_A f_1 d\mu, \int_A f_2 d\mu, \cdots\right)$$

where $f_i \in CD(X)$, and $CD(X)$ is a countable dense subset of $C_+(X)$ mentioned before. By theorem 3.2, we know that $\mu_1 = \mu_2$ if $\int_A f_i d\mu_1 = \int_A f_i d\mu_2$

It shows that the mapping H is a one - to - one mapping form $FM(X)$ to $H(FM(X)) \subset R^\infty$. At this time, we may define a metric δ on R^∞.

$$\delta: R^\infty \times R^\infty \to [0,1]$$
$$(x, y) \to \delta(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \min\{1, |x_i - y_i|\}$$

where $x = (x_1, x_2, \cdots), y = (y_1, y_2, \cdots) \in R^\infty$

By [9], evidently, H is a homeomorphic mapping from $(FM(X), \rho)$ to the subspace $H(FM(X))$ of (R^∞, δ). Since (R^∞, δ) is a separable metric space, consequently, $(FM(X), \rho)$ is also separable. The proof is completed.

Corollary 4.2′ If X is a compact and separable metric space, then $(FM(X), \rho)$ is complete metric space.

References

[1] D. Dubo is and H. Prade, Fuzzy Sets and Systems - Theory and Applications (Academic press, New York, 1980).

[2] D. Ralescu and G. Adams, The fuzzy integral, J. Math. Anal. Appl. 75(1980)562 - 570.

[3] Zhenyuan Wang, The autocontinuity of set - function and the fuzzy integral, J. Math. Anal. Appl. 99(1984)195 - 218.

[4] Deli Zhang and Caimei Guo, On the convergence of sequences of fuzzy measures and generalized convergence theorems of fuzzy integrals, Fuzzy Sets and Systems 72(1995)349 - 356.

[5] H. Roman - Flores etc., On the lever - continuity of fuzzy integrals, Fuzzy Sets and Systems 80 (1996)339 - 344.

[6] M. Sugeno, Theory of fuzzy integrals and its applications, ph. D. Disseration, Tokyo In stitute of Technology 1974).

[7] P. R. Halmos, Measure Theory (Van Nostrand, New York, 1956)

[8] Zhongshan University. Measure and probability Basis (Guangdong Science and Technique Press. 1981.7)

[9] Jiang Zehan, Topology Basis. (Shanghai Science and Technique press. 1978)

[10] Zhang Wenxiu, Basis of Fuzzy Mathematics (Xian Jiaotong Univ. Press. Xian, 1984)