FURTHER DISCUSSIONS ON GENERALIZED FUZZY INTEGRALS ON FUZZY SETS(II): Fuzzy-valued fuzzy measures and generalized fuzzy integrals on fuzzy sets

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Abstrac. This paper is a continuous work of [8], the concept of fuzzy-valued fuzzy measures on fuzzy sets is introduced at first, then basing on the generalized fuzzy integral on fuzzy sets given by Wu[5], the generalized fuzzy integral of fuzzy-valued functions with respect to fuzzy-valued fuzzy measures on fuzzy sets is defined, the properties and convergence theorems are shown. All these are the extension of [8].

Keywords. Fuzzy measure on fuzzy sets, generalized fuzzy integral on fuzzy sets, fuzzy-valued function, fuzzy-valued fuzzy measure on fuzzy sets.

1. Introduction

Since Sugeno[4] introduce the concepts of fuzzy measures and fuzzy integrals, the theory has been made much deeper by Ralescu and Adams[5], and many others. Especially, the generalized fuzzy integral on fuzzy sets introduced by Wu[5] is a much wider one, and it keeps all the results corresponding Sugeno's. In ref. [8], we have established the theory of generalized fuzzy integrals of fuzzy-valued functions, it is a good extension of generalized fuzzy integrals of point-valued functions. This paper is in a futher way to generalize it, we will build up the theory of generalized fuzzy integrals of fuzzy-valued functions with respect to fuzzy-valued fuzzy measures on fuzzy sets.

In the remaider of this paper, R^+ denotes the interval $[0, +\infty]$, $I(R^+)$ denotes the interval number set on R^+, $\tilde{R}^+$ denotes the fuzzy number set on R^+. X is an arbitrary fixed set, $\tilde{\mathscr{A}}$ is fuzzy a σ-algebra formed by the fuzzy subsets of X, $(X, \tilde{\mathscr{A}})$ is the measurable space. Let $\tilde{F}(X)$ denotes the set of all $\tilde{\mathscr{A}}$-measurable fuzzy-valued functions from X to $\tilde{R}^+$, and $\tilde{M}(X)$ denotes the set of all fuzzy measures from $\tilde{\mathscr{A}}$ to $\tilde{R}^+$. Other notations and concepts which are not mentioned, can be found in [5,6,7,8].

2. Fuzzy-valued measures on fuzzy sets

Definition 2.1. A mapping $\bar{\mu}$ (resp. $\tilde{\mu}$): $\tilde{\mathscr{A}} \to I(R^+)$ (resp. $\tilde{R}^+$) is said to be an interval-valued (resp. fuzzy-valued) fuzzy measure if it satisfies the following conditions:

(i) $\bar{\mu}(\Phi) = 0$ (resp. $\tilde{\mu}(\Phi) = \tilde{O}$), where $\tilde{O}(r) = \begin{cases} 1, & r = 0 \\ 0, & r \neq 0. \end{cases}$;

(ii) $\tilde{A} \subset \tilde{B}$ implies $\bar{\mu}(\tilde{A}) \leqslant \bar{\mu}(\tilde{B})$ (resp. $\tilde{\mu}(\tilde{A}) \leqslant \tilde{\mu}(\tilde{B})$);

(iii) $\tilde{A}n \uparrow \tilde{A}$ implies $\tilde{\mu}(\tilde{A}n) \uparrow \tilde{\mu}(\tilde{A})$ (resp. $\tilde{\mu}(\tilde{A}n) \overset{s}{\to} \tilde{\mu}(\tilde{A})$);

(iv) $\tilde{A}n \downarrow \tilde{A}$ and there exists a n_0, s.t. $\bar{\mu}(\tilde{A}n_0) < \infty$ (resp. $\tilde{\mu}(\tilde{A}n_0) < \infty$), implies $\bar{\mu}(\tilde{A}n) \downarrow \bar{\mu}(\tilde{A})$ (resp. $\tilde{\mu}(\tilde{A}_n) \overset{s}{\to} \tilde{\mu}(A)$).

Remark: "$\to$, $\overset{s}{\to}$" can be found in [6].

In the following, we use $\overline{M}(X)$ (resp. $\widetilde{M}(X)$) to denote the set of all interval-valued (resp. fuzzy-valued) fuzzy measures. The relationship of fuzzy measures, interval-valued fuzzy measures and fuzzy-valued fuzzy measures are shown as:

Lemma 2.1. $\bar{\mu} \in \overline{M}(X)$ iff μ^-, $\mu^+ \in M(X)$, where $\mu^-(A) = \inf\bar{\mu}(A)$, $\mu^+(A) = \sup\bar{\mu}(A)$.

Lemma 2.2. If $\tilde{\mu} \in \widetilde{M}(X)$, then $\bar{\mu}_\lambda \in \overline{M}(X)$ for every $\lambda \in (0, 1]$. Where $\bar{\mu}_\lambda(\tilde{A}) = (\tilde{\mu}(\tilde{A}))_\lambda$, and $\bar{\mu}_{\lambda 1} \supset \bar{\mu}_{\lambda 2}$ (i.e. $\bar{\mu}_{\lambda 1}(\tilde{A}) \supset \bar{\mu}_{\lambda 2}(\tilde{A})$ for all $\tilde{A} \in \tilde{\mathscr{A}}$) holds for $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant 1$.

3. Generalized fuzzy integrals of fuzzy-valued functions with respect to fuzzy-valued fuzzy measures on fuzzy sets.

Definition 3.1. Let $\bar{f}$ be a $\tilde{\mathscr{A}}$-measurable interval-valued function, $\tilde{A} \in \tilde{\mathscr{A}}$, and $\bar{\mu} \in \overline{M}(X)$. Then the (G) fuzzy integral of $\bar{f}$ over $\tilde{A}$ with respect to $\bar{\mu}$ is defined as,

$$\int_{\tilde{A}} \bar{f} d\bar{\mu} = \left[\int_{\tilde{A}} f^- d\mu^-, \int_{\tilde{A}} f^+ d\mu^+ \right].$$

Definition 3.2. Let $\tilde{f} \in \tilde{F}(X)$, $\tilde{A} \in \tilde{\mathscr{A}}$ and $\tilde{\mu} \in \widetilde{M}(X)$. Then the (G) fuzzy integral of $\tilde{f}$ over $\tilde{A}$ with respect $\tilde{\mu}$ is defined as

$$\left(\int_{\tilde{A}} \tilde{f} d\tilde{\mu} \right)(r) = \operatorname{Sup}\{\lambda \in (0, 1]: r \in \int_{\tilde{A}} \bar{f}_\lambda d\bar{\mu}_\lambda\}$$

Theorem 3.1. Let $\tilde{f} \in \tilde{F}(X)$, $\tilde{\mu} \in \widetilde{M}(X)$ and $\tilde{\mu}(\tilde{A}) < \infty$. Then $\int_{\tilde{A}} \tilde{f} d\tilde{\mu} \in \tilde{R}^+$, and

$$\left(\int_{\tilde{A}} \tilde{f} d\tilde{\mu} \right)_\lambda = \int_{\tilde{A}} \bar{f}_\lambda d\bar{\mu}_\lambda, \ (\lambda \in (0, 1]) \qquad (4.1)$$

Theorem 3.2. (G) fuzzy integrals of fuzzy-valued functions with respect to fuzzy-valued fuzzy measures on fuzzy sets have following properties:

(i) $\tilde{f}_1 \leqslant \tilde{f}_2$ implies $\int_{\tilde{A}} \tilde{f}_1 d\tilde{\mu} \leqslant \int_{\tilde{A}} \tilde{f}_2 d\tilde{\mu}$,

(ii) $\tilde{A} \subset \tilde{B}$ implies $\int_{\tilde{A}} \tilde{f} d\tilde{\mu} \leqslant \int_{\tilde{B}} \tilde{f} d\tilde{\mu}$

(iii) $\tilde{\mu}_1 \leqslant \tilde{\mu}_2$ implies $\int_{\tilde{A}} \tilde{f} d\tilde{\mu}_1 \leqslant \int_{\tilde{A}} \tilde{f} d\tilde{\mu}_2$,

(iv) $\tilde{\mu}(\tilde{A}) = \tilde{O}$ implies $\int_{\tilde{A}} \tilde{f} d\tilde{\mu} = \tilde{O}$, where $\tilde{O}(r) = \begin{cases} 1, & r = 0, \\ 0, & r \neq 0. \end{cases}$

(v) $\int_{A} \tilde{r} d\tilde{\mu} = S(\tilde{r}, \tilde{\mu}(\tilde{A}))$, where $\tilde{r} \in \tilde{R}^+$ and we define $S(\bar{r}, \bar{p}) = [S(r^-, p^-), S(r^+, p^+)]$, [S

$(\tilde{r}, \tilde{p})]_\lambda = S(\bar{r}_\lambda, \bar{p}_\lambda)$, for $\bar{r}, \bar{p} \in I(R^+)$, $\tilde{r}, \tilde{p} \in \tilde{R}^+$.

(vi) $\int_{\tilde{A}} (\tilde{r} \vee \tilde{f}) d\tilde{\mu} = \int_{\tilde{A}} \tilde{r} d\tilde{\mu} \vee \int_{\tilde{A}} \tilde{f} d\tilde{\mu}$, where $\tilde{r} \in \tilde{R}^+$.

Theorem 3.3. Let $\{\tilde{f}_n, (n \geq 1), \tilde{f}\} \subset \tilde{F}(X)$, $\{\tilde{\mu}_n (n \geq 1), \tilde{\mu}\} \subset \tilde{M}(X)$. If $\tilde{f}_n \uparrow \tilde{f}, \tilde{\mu}_n \uparrow \tilde{\mu}$, and $\tilde{\mu}(\tilde{A}) < \infty$, then $\int_{\tilde{A}} \tilde{f}_n d\tilde{\mu}_n \uparrow \int_{\tilde{A}} \tilde{f} d\tilde{\mu}$

Theorem 3.4. Let $\{\tilde{f}_n (n \geq 1), \tilde{f}\} \subset \tilde{F}(X)$, $\{\tilde{\mu}_n (n \geq 1), \tilde{\mu}\} \subset \tilde{M}(X)$. If $\tilde{f}_n \downarrow \tilde{f}, \tilde{\mu}_n \downarrow \tilde{\mu}$, and there exists a n_0, s.t. $\tilde{\mu}_{n_0}(\tilde{A}) < +\infty$, then $\int_{\tilde{A}} \tilde{f}_n d\tilde{\mu}_n \downarrow \int_{\tilde{A}} \tilde{f} d\tilde{\mu}$

Corollary 3.1. Let $\{\tilde{f}_n (n \geq 1), \tilde{f}\} \subset \tilde{F}(X)$, $\tilde{\mu} \in \tilde{M}(X)$, and $\tilde{\mu}(\tilde{A}) < \infty$. If $\tilde{f}_n \uparrow \tilde{f}$ or $\tilde{f}_n \downarrow \tilde{f}$, then $\int_{\tilde{A}} \tilde{f}_n d\tilde{\mu} \to \int_{\tilde{A}} \tilde{f} d\tilde{\mu}$.

Corollary 3.2. Let $\tilde{f} \in \tilde{F}(X)$, $\{\tilde{\mu}_n (n \geq 1), \tilde{\mu}\} \subset \tilde{M}(X)$.

(i) If $\tilde{\mu}_n \uparrow \tilde{\mu}$, and $\tilde{\mu}(\tilde{A}) < +\infty$, then $\int_{\tilde{A}} \tilde{f} d\tilde{\mu}_n \uparrow \int_{\tilde{A}} \tilde{f} d\tilde{\mu}$;

(ii) If $\tilde{\mu}_n \downarrow \tilde{\mu}$, and there exists a n_0, s.t. $\tilde{\mu}_{n_0}(\tilde{A}) < +\infty$, then $\int_{\tilde{A}} \tilde{f} d\tilde{\mu}_n \downarrow \int_{\tilde{A}} \tilde{f} d\tilde{\mu}$.

Theorem 3.5. Let $\{\tilde{f}_n\} \subset \tilde{F}(X)$, $\{\tilde{\mu}_n\} \subset \tilde{M}(X)$. If $\{\bigwedge_{k=n}^{\infty} \tilde{\mu}_n (n \geq 1), \bigvee_{k=n}^{\infty} \tilde{\mu}_n (n \geq 1)$, liminf $\tilde{\mu}_n$, limsup $\tilde{\mu}_n\} \subset \tilde{M}(X)$, and $\tilde{\mu}_n(\tilde{A}) < +\infty (n \geq 1)$, then

(i) $\int_{\tilde{A}} (liminf\, \tilde{f}_n) d(liminf \tilde{\mu}_n) \leq \liminf_{n \to \infty} \int_{\tilde{A}} \tilde{f}_n d\tilde{\mu}_n$,

(ii) $limsup \int_{\tilde{A}} \tilde{f}_n d\tilde{\mu}_n \leq \int_{\tilde{A}} (limsup \tilde{f}_n) d(limsup \tilde{\mu}_n)$.

Corollary 3.3. Let $\{\tilde{f}_n\} \subset \tilde{F}(X)$, $\tilde{\mu} \in \tilde{M}(\tilde{A})$, and $\tilde{\mu}(\tilde{A}) < \infty$. Then

(i) $\int_{\tilde{A}} (liminf \tilde{f}_n) d\tilde{\mu} \leq liminf \int_{\tilde{A}} \tilde{f}_n \tilde{\mu}$

(ii) $limsup \int_{\tilde{A}} \tilde{f}_n d\tilde{\mu} \leq \int_{\tilde{A}} (limsup \tilde{f}_n) d\tilde{\mu}$.

Concluding remark.

Up to here, we have built up a theory of generalized fuzzy integrals on fuzzy sets, it is the most general one as far as we know. The theory is based on fuzzy numbers in[2], of course, we can establish the similar theory based on other concepts of fuzzy numbers. That will be given in a subsquent paper.

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