FOURIER ANALYSIS ON FUZZY SETS
Part One

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Abstract

A universe G and a family Γ of fuzzy sets in G are equipped with appropriate structures in order to make them locally compact topological groups which are dual to each other. These structures will, ofcourse, provide us with powerful mathematical tools of Fourier analysis to manipulate fuzzy propositions. This approach seems to open a new horizon in the field. *Part One* of this study concerns with the construction of the basic mathematical structures on G, and on Γ.

The Universe as a LCA Group

Let X be a universe, whose generic elements are denoted by x, y, etc. A fuzzy set is simply a function from X into the interval $[0,1]$. Let F be a set (more precisely a family) of fuzzy sets in X. Then we have

$$\gamma \in F \Rightarrow \gamma : X \to [0,1]. \tag{1}$$

Fuzzy sets are rich enough to assume further that F separates points on X, i.e.,to every pair of distinct points $x_1, x_2 \in X$ there corresponds an element $\gamma \in F$ such that $\gamma(x_1) \neq \gamma(x_2)$.

Since, the set F of fuzzy sets in X is a well-defined classical set, one may also consider another set (family) $\tilde{F}$ of fuzzy sets in F. As in (1) we write

$$x \in \tilde{F} \Rightarrow x : F \to [0,1] \tag{2}$$

Throughout the paper the unit interval $[0,1]$ will represent the additive group $\Re$ *mod* 1, which is equivalent, as a LCA (locally compact abelian) topological group, to the unit circle $T = \{z : \ z \in \mathbf{C}, \ |z| = 1\}$ where $\mathbf{C}$ stands for the complex plane.

We define a binary operation $\oplus$ by the relation

$$(p \oplus q)(x) = (p(x) + q(x)) \; mod \; 1, \; p, q \in F, \; x \in \tilde{F}. \tag{3}$$

and, similarly, the operation $\tilde{\oplus}$ by the relation

$$(x \tilde{\oplus} y)(p) = (p(x) + p(y)) \; mod \; 1, \; p \in F, \; x, y \in \tilde{F}. \tag{4}$$

Notice that neither F nor $\tilde{F}$ need to be closed to the operations $\oplus$ and $\tilde{\oplus}$, respectively. These deficits will be removed sooner.

For our future purpose it will be more convenient to replace $[0, 1]$, i.e., the additive group $\Re \; mod \; 1$ with the multiplicative group T.

Assuming the above replacement $[0, 1]$ with T, for a fixed $\gamma \in F$, while x varies in $\tilde{F}$, the symbol (x, γ) will be interpreted as "γ is a function from $\tilde{F}$ into T." Likewise, for a fixed $x \in \tilde{F}$, while γ varies in F, the symbol (x, γ) will also be interpreted as "x is a function from F into T."

In view of this duality between F and $\tilde{F}$, we may interchangebly use the symbols $x(p) = (x, p) = p(x), \quad p \in F, \; x \in \tilde{F}$ to make it adequate to its context.

Having done these replacements, the relations (1) to (4) will equivalently be replaced by

$$p \in F \Rightarrow p : X \rightarrow T\} \tag{5}$$

$$x \in \tilde{F} \Rightarrow x : F \rightarrow T\} \tag{6}$$

$$(p \oplus q)(x) = (x, p)(x, q), \quad p, q \in F, \; x \in \tilde{F} \tag{7}$$

and

$$(x \tilde{\oplus} y)(p) = (x, p)(y, p), \quad p \in F, \; x, y \in \tilde{F} \tag{8}$$

in their respective order.

We note that, by the duality quoted above, each $x \in X$ can be viewed as a function from F into T. Therefore, it is legitimate to assume that $\tilde{F}$ is chosen to be the set X.

Lemma 1 *X separates points of F.*

Let us now consider the semigroup Γ generated by F and the semigroup G generated by $\tilde{F} = X$ with respect to the binary operations defined in (7) and (8), respectively. Then the following two theorems can be constructed by simple manipulations.

Theorem 1 $(\Gamma, \oplus)$ *is an Abelian group.*

Theorem 2 $(G, \tilde{\oplus})$ *is an abelian group.*

For the sake of simplicity, we shall use the symbol $+$ in place of both operations $\oplus$ and $\tilde{\oplus}$. Thus 0 will stand for the units of the groups G and Γ. Their meanings will always be clear from their context. We also note that 1 will denote the unit of the multiplicative group T. Moreover, $\mathcal{B}(z)$ will denote a base of neighbourhoods of z in a topological space. The following lemma follows from the definition.

Lemma 2 *For all* $\gamma \in \Gamma$ *and all* $x \in G$ *we have* $(-x, \gamma) = \overline{(x, \gamma)} = (x, -\gamma)$.

A Locally Compact Topology on the Universe

We shall endow G with a topology with respect to which G will be a LCA topological group.

Let $\mathcal{S}$ be the set of all subsets of G of the form $p^{-1}(V)$ ($p \in \Gamma$ and V is open in T), and let $\mathcal{B}$ be the set of all finite intersections of sets of $\mathcal{S}$. Then, $\mathcal{B}$ is a base of a topology τ on G, which is called the initial (projective) topology on G for the family $p \in \Gamma$. τ is the coarsest topology on G for which all the mappings $p \in \Gamma$ are continuous. Then we have the following results.

Lemma 3 *Let* V *be a symmetric neighborhood of* 1 *in* T *and let* p *be an element of* Γ. *Then the set*

$$U_{V,p} = \{x \in G : (x, p) \in V\} \tag{9}$$

is a symmetric neighborhood of 0, *in* G.

Lemma 4 *Let* V_1 *and* V_2 *be two symmetric neighborhoods of* 1 *in* T *with the property that* $V_1 \subset V_2$, *and let* $p \in \Gamma$. *Then we have* $U_{V_1,p} \subset U_{V_2,p}$.

Lemma 5 *The family of all the sets* $U_{V,p}$, *where* V *runs through a base of symmetric neighborhoods of the unit* 1 *in* T *and where* $p \in \Gamma$ *forms a base of symmetric neighborhoods of* 0 *in* G.

Lemma 6 *If* V *runs through a base of symmetric neighbourhood of* 1 *in* T, *and if* p *runs through* Γ, *then the family of the sets* $\{x + U_{V,p}\}$ *forms a base of neighbourhoods of* x *for any* $x \in G$.

Lemma 7 *Given any neighbourhood W of 0 in G, there exist an element p in Γ and a symmetric neighbourhood V of 1 in T such that $U_{V,p} + U_{V,p} \subset W$*

Lemma 8 *(G, τ) is a Hausdorff topological space.*

Lemma 9 *p is an open mapping from G into T for each $p \in \Gamma$.*

Lemma 10 *If V is a compact neighbourhood of 1 in T, and if $p \in \Gamma$, then $U_{V,p}$ is compact in G.*

Theorem 3 *(G, τ) is locally compact.*

Theorem 4 *$(G, +, \tau)$ is a LCA topological group.*

Definition 1 *A complex function γ on a LCA group G is called a character of G if $|\gamma(x)| = 1$ for all $x \in G$ and if the functional equation*

$$\gamma(x + y) = \gamma(x)\gamma(y) \quad (x, y) \in G \tag{10}$$

is satisfied.

Let us denote the set of all continuous characters γ of G by Ψ. $(\Psi, +)$ is an abelian group if addition is defined by

$$(\gamma_1 + \gamma_2)(x) = \gamma_1(x)\gamma_2(x) \quad (x \in G;\ \gamma_1, \gamma_2 \in \Psi). \tag{11}$$

$(\Psi, +)$ is called the dual group of G.

Theorem 5 *$(\Gamma, +)$ is the dual of the LCA topological group $(G, +, \tau)$.*

Notice that, the dual group Γ is nothing but a family of fuzzy sets in the group G, generated by the universe X.

Haar Measure on the Universe

We shall assume the basic theory of Haar measure, convolution and Fourier transformations. We only recall that on every LCA group G there exists a non-negative regular measure m, the so called *Haar measure* of G, which is not identically 0 and which is *translation invariant*. That is to say,

$$m(x + B) = m(B) \tag{12}$$

for every $x \in G$ and every Borel set B in G.

The Fourier transform of a function $f \in L^1(G)$ is defined by

$$\hat{f}(\gamma) = \int_G f(x)(-x,\gamma)dm(x) \quad (\gamma \in \Gamma) \tag{13}$$

The set of all functions $\hat{f}$ so obtained will be denoted by $A(\Gamma)$. $\hat{f}$ is the Gelfand transform of f. Thus if we equip Γ with the weak topology induced by $A(\Gamma)$, the basic facts of Gelfand theory concerning Banach algebras can be used to characterize Fourier transforms.

Fuzzy Sets as the Dual Group of the Universe

We shall endow the dual group Γ with the projective topology Υ for the family $x \in G$. Υ is the coarsest topology on Γ for which all the mappings

$$x : \Gamma \rightarrow T \quad (x \in G) \tag{14}$$

are continuous. The following two results are the counterparts to *Lemma 3* and *Theorem 4*.

Lemma 11 *The family of the sets*

$$S_{V,x} = \{\gamma \in \Gamma \ : \ (x,\gamma) \in V\}, \tag{15}$$

where V runs through a base of symmetric neighbourhoods of the unit 1 *in T and where $x \in G$ forms a base of symmetric neighbourhoods of the unit* 0 *in* Γ.

Theorem 6 $(\Gamma, +, \Upsilon)$ *is a LCA topological group.*

Theorem 7 *The topology Υ is equal to the Gelfand topology on Γ.*

Conclusion

The theorems stated above will enable us to apply the tools of abstract harmonic analysis to the dual pair G and Γ. The remaining of this study will mainly deal with the fourier expansions of propositions.

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