36

The Fuzzy Derivative a la Caratheodory

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As we known , the fuzzy differentials is a very important and difficult subject in fuzzy mathematics. In this note, we use caratheodory's derivative notion to define the fuzzy derivative. This method is different from [1]. At the same time, We also give a few basic properties of fuzzy derivative.

Definition 1. Let E be a vector space over the field K of real or complex numbers, (E, T) be a fuzzy topological space, if the two mappings

(i) $\sigma : E \times E \to E, (x, y) \to x + y$

(ii) $\pi : K \times E \to E, (\alpha, x) \to \alpha x$

where K is the induced fuzzy topology of the usual norm, are fuzzy continuous. Then (E, T) is said to be a fuzzy topological vector space over the field K.

Definition 2 (Caratheodory). Let f : $(a, b) \subseteq R \to R, c \in (a, b)$, the function f is said to be differentiable at the point $c \in (a, b)$ if there exists a function ϕ_c that is continuous at $x = c$ and satisfies the relation $f(x) - f(c) = \phi_c(x)(x - c)$ for all $x \in$ (a, b).

We will usually write $\phi(x)$ instead of $\phi_c(x)$, since there seems to be little chance of confusion, but we must remember that the function ϕ depends on the point c. Geometrically, of course, when $x \neq c, \phi_c(x)$ is the slope of the secant line through the points $(x, f(x))$ and $(c, f(c))$. This alternative definition emphasizes the fact that the slopes of the secant lines , by way of which we initially arrive at the tangent line, approach the tangent line in a continuous manner, we rarely state this important fact explicitly, but we should.

This definition can be used as is for complex-valued functions of a complex variable and , in fact, even for function of several variables [2,3].

For simplicity, we only consider real-valued functions of one variable.

Definition 3. Let R be the field of real numbers and (R, T) be a fuzzy topological vector space over the field R . $f : R \rightarrow R$. $a \in R$, the function f is said to be fuzzy differentiable at the point a if there is a function ϕ that is fuzzy continuous at $x = a$ and have $f(x) - f(a) = \phi(x)(x - a)$ for all $x \in R$. $\phi(a)$ is said to be the fuzzy derivative of f at a and denote $f'(a) = \phi(a)$.

Now, we give a few basic properties of fuzzy derivative.

Theorem 1 (Chain Rule). If f is fuzzy differentiable at the point a and g is fuzzy differentiable at the point $f(a)$, then h = gof is also fuzzy differentiable at the point a and $h'(a) = g'(f(a))f'(a)$.

Proof. By the conditions, thene are φ and ψ such that $f(x) - f(a) = \varphi(x)(x - a)$ and $g(y) - g(f(a)) = \psi(y)(y - f(a))$.

Here $\varphi(x)$ is fuzzy continuous at the point a and ψ is fuzzy continuous at the point $f(a)$. Hence, we have $h(x) - h(a) = g(f(x)) - g(f(a)) = \phi[f(x)][f(x) - f(a)] = \psi(f(x))\phi(x)(x - a)$. Since $(\psi o f(x))\varphi$ (x) is fuzzy continuous at a and $(\psi o f(x))\varphi(x)|_{x=a} = \psi(f(a))\varphi(a)$, the proof is complete.

Theorem 2 (Critical Point Theorem). If f is fuzzy differentiable at the point a and f(a) is an extreme value then a is a critical point ($i.e.$, $f'(a) = 0$).

Proof . We only prove the Theorem for $f(a)$ is a maximum. By the conditions, there exists a function ϕ that is fuzzy continuous at a and $f(x) - f(a) = \varphi(x)(x - a)$.

Since f(a) is a maximum. So we can obtain an $\varepsilon_0 > 0$ such that when $x \in (a - \varepsilon_0, a), \varphi(x) > 0$; when $x \in (a, a + \varepsilon_0), \varphi(x) < 0$.

Now , we prove that $\varphi(a) = 0$. For otherwise, we may suppose that $\varphi(a) >$

0. Taking $\varepsilon > 0$ such that $\varphi(a) - \varepsilon > 0$. Since the characteristic function of open set is lower semi-continuous, so every open set of (R, $|\cdot|$) is also the fuzzy open set of (R, T). This shows that $\varphi^{-1}((\varphi(a) - \varepsilon, \varphi(a) + \varepsilon))$ is fuzzy open set of (R, T). It is easily to prove that the membership function of $\varphi^{-1}((\varphi(a)-\varepsilon, \varphi(a)+\varepsilon))$ is the characteristic function of $\varphi^{-1}((\varphi(a)-\varepsilon, \varphi(a)+\varepsilon))$. Since $a \in \varphi^{-1}((\varphi(a)-\varepsilon, \varphi(a)+\varepsilon))$, by the definition of fuzzy topological space (R, T) we know that the characteristic function of $\varphi^{-1}((\varphi(a) - \varepsilon, \varphi(a) + \varepsilon))$ is $|\cdot|$ - lower semi-continuous. So there exists $\varepsilon_1 > 0$ such that $(a - \varepsilon_1, a + \varepsilon_1) \subseteq \varphi^{-1}((\varphi(a) - \varepsilon, \varphi(a) + \varepsilon))$. That is $\varphi((a - \varepsilon_1, a + \varepsilon_1)) \subseteq (\varphi(a) - \varepsilon, \varphi(a) + \varepsilon)$. This is a contradiction. So $\varphi(a) = 0$. Similar, we have

Theorem 3 (Inverse Function Theorem) . Let f be fuzzy continuous and strictly monotonic on R and f be fuzzy differentiable at the point a , if $f'(a) \neq 0$, then $g = f^{-1}$ is fuzzy differentiable at the point $d =. f(a)$ and $g'(d) = [f'(a)]^{-1}$.

References

[1] M. Ferraro, D.H. Foster, Differentiation of Fuzzy Continuous Mappings on Fuzzy Topological Vector Spaces, J. Math. Anal. Appl. 2(1987), 589-601.

[2] S. Kuhn, The Derivative *ala* Caratheodory, Amer. Math. Monthly, 98(1991), 40-44.

[3] A. G. Ernesto, D.G. Cesar, Frechet VS, Caratheodory, Amer. Math. Monthly, 101(1994), 332-338.