On Intuitionistic Fuzzy Subgroups and Their Products

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Abstract: The purpose of this paper is to introduce intuitionistic fuzzy subgroups using the concept of intuitionistic fuzzy set developed by Atanassov [3,4,5].

Keywords: Intuitionistic fuzzy set; intuitionistic fuzzy pair; intuitionistic fuzzy subgroupoid; intuitionistic fuzzy subgroup; intuitionistic fuzzy norm; fuzzy product.

1. Introduction

The theory of fuzzy subgroups was initiated by Rosenfeld [12], Negoita-Ralescu [11], Anthony-Sherwood [1,2] and so many contributions were made on these main directions. On the other hand, intuitionistic fuzzy sets was introduced by Atanasssov [3,4,5]. In this present paper we shall give an introduction to intuitionistic fuzzy subgroups.

2. Preliminaries

After the introduction of the concept of a fuzzy set by Zadeh [14], Atanassov [3,4] has introduced the concept of intuitionistic fuzzy set (IFS for short):

Definition 2.1. [3,4] Let X be a nonempty fixed set. An intuitionistic fuzzy set A is an object having the form

$$A=\{ \langle x,\mu_A(x),\gamma_A(x)\rangle : x\in X \}$$

where the functions $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) of each element $x\in X$ to the set A, respectively, and $0\le\mu_A(x)+\gamma_A(x)\le 1$ for each $x\in X$. For the sake of simplicity, we shall use the symbol $A=\langle x,\mu_A,\gamma_A\rangle$ for the IFS $A=\{\langle x,\mu_A(x),\gamma_A(x)\rangle: x\in X\}$.

Every fuzzy set A on a nonempty set X is obviously an IFS having the form $A=\{ \langle x,\mu_A(x),1-\mu_A(x)\rangle : x\in X \}$ [ ]. The definitions and properties of inclusion, equality, intersection, union, preimage,and image regarding intuitionistic fuzzy sets and intuitionistic fuzzy pairs can be found in [4,5,6].

Definition 2.2. [4,5,6] Let X be a nonempty set, and the IFS A be

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in the form $A=\langle x,\mu_A,\gamma_A\rangle$. Then

(a) $[]A=\langle x,\mu_A,1-\mu_A\rangle$; (b) $\langle\rangle A=\langle x,1-\gamma_A,\gamma_A\rangle$.

Definition 2.3. [cf. 10] An intuitionistic fuzzy norm (IFN for short) in $[0,1]\times[0,1]$ is a couple $\langle T,S\rangle$ of two mappings

$$T,\ S : [0,1] \times [0,1] \rightarrow [0,1]$$

satisfying the following conditions:

(1) $T(x,1)=x$, $T(x,0)=0$, $S(x,1)=1$, $S(x,0)=x$ for all $x\in[0,1]$ (BOUNDARY CONDITIONS);

(2) $T(x,y)\le T(z,t)$, $S(x,y)\le S(z,t)$ for all $x,y,z,t\in[0,1]$ satisfying $x\le z$ and $y\le t$ (MONOTONY);

(3) $T(x,y)=T(y,x)$ and $S(x,y)=S(y,x)$ for all $x,y\in[0,1]$ (COMMUTATIVITY);

(4) $T(T(x,y),z)=T(x,T(y,z))$ and $S(S(x,y),z)=S(x,S(y,z))$ for all $x,y,z\in[0,1]$ (ASSOCIATIVITY);

(5) $T(x,y)+S(z,t)\le 1$ for all $x,y,z,t\in[0,1]$ satisfying $x+z\le 1$ and $y+t\le 1$ (CONNECTION).

3. Intuitionistic fuzzy subgroups

First we shall present the concept of intuitionistic fuzzy subgroup in the sense of Rosenfelf [12]. Here $A(x)$ can be thought as an intuitionistic fuzzy pair $A(x)=\langle\mu_A(x),\gamma_A(x)\rangle$.

Definition 3.1. [cf. 12] Let $(G,.)$ be a groupoid. An intuitionistic fuzzy subgroupoid in G is an IFS $A=\langle g,\mu_A,\gamma_A\rangle$ satisfying the condition $A(x.y)\ge A(x)\wedge A(y)$ for all $x,y\in G$.

Remarks 3.2. (1) The condition $A(x.y)\ge A(x)\wedge A(y)$ means exactly that

$$\mu_A(x.y)\ge\mu_A(x)\wedge\mu_A(y) \quad\text{and}\quad \gamma_A(x.y)\le\gamma_A(x)\vee\gamma_A(y),$$

for all $x,y\in G$.

(2) If A is an intuitionistic fuzzy subgroupoid in G, then μ_A and $1-\gamma_A$ are fuzzy subgroupoids in G in the sense of Rosenfeld [12], too, and we also have $\mu_A\le 1-\gamma_A$.

(3) If μ_A is a fuzzy subgroupoid in the sense of Rosenfeld [12], then the IFS's $A=\langle g,\mu_A,1-\mu_A\rangle$, $A_1=\langle g,\mu_A,0\rangle$ and $A_2=\langle g,0,1-\mu_A\rangle$ are intuitionistic fuzzy subgroupoids in the sense of Definition 3.1, too.

(4) If A is an intuitionistic fuzzy subgroupoid in G, then $[]A$ and $\langle\rangle A$ are intuitionistic fuzzy subgroupoids in G, too.

Definition 3.3. [cf. 12] Let $(G,.)$ be a group. An intuitionistic fuzzy subgroup in G (IFG for short) is an intuitionistic fuzzy subgoupoid $A=\langle g,\mu_A,\gamma_A\rangle$ satisfying the condition $A(x^{-1})\geq A(x)$ for all $x\in G$.

Corollary 3.4. If $A=\langle g,\mu_A,\gamma_A\rangle$ is an IFG in G, then we have $A(x^{-1})=A(x)$ and $A(x)\leq A(e)$ for all $x\in G$.

Corollary 3.5. If $A=\langle g,\mu_A,\gamma_A\rangle$ is an IFG in G, then the IFS A^{-1} defined by
$A^{-1}=\langle g,\mu_{A^{-1}},\gamma_{A^{-1}}\rangle$, where $\mu_{A^{-1}}(x)=\mu_A(x^{-1})$ and $\gamma_{A^{-1}}(x)=\gamma_A(x^{-1})$.
is also an IFG in G.

Example 3.6. Let $G=\{e,x_1,x_2,x_3,x_4,x_5,x_6,x_7\}$ be the quaternion group given by

$\cdot$	e	x_1	x_2	x_3	x_4	x_5	x_6	x_7
e	e	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_1	x_1	x_4	x_3	x_6	x_5	e	x_7	x_2
x_2	x_2	x_7	x_4	x_1	x_6	x_3	e	x_5
x_3	x_3	x_2	x_5	x_4	x_7	x_6	x_1	e
x_4	x_4	x_5	x_6	x_7	e	x_1	x_2	x_3
x_5	x_5	e	x_7	x_2	x_1	x_4	x_3	x_6
x_6	x_6	x_3	e	x_5	x_2	x_7	x_4	x_2
x_7	x_7	x_6	x_1	e	x_3	x_2	x_5	x_4

Then the IFS

$$A=\langle x,\ \left(\frac{e}{0.6},\frac{x_1}{0.2},\frac{x_2}{0.3},\frac{x_3}{0.2},\frac{x_4}{0.5},\frac{x_5}{0.2},\frac{x_6}{0.3},\frac{x_7}{0.2}\right),$$

$$\left(\frac{e}{0.3},\frac{x_1}{0.7},\frac{x_2}{0.6},\frac{x_3}{0.7},\frac{x_4}{0.4},\frac{x_5}{0.7},\frac{x_6}{0.6},\frac{x_7}{0.7}\right)\rangle$$

is an IFG on G.

Definition 3.7. [cf. 8] Let A be an IFG on G and $\langle\alpha,\beta\rangle$ be an intuitionistic fuzzy pair such that $A(e)\geq\langle\alpha,\beta\rangle$. Then the set

$$A_{\langle\alpha,\beta\rangle}=\{\ x\in G\ :\ A(x)\geq\langle\alpha,\beta\rangle\ \}$$

is called the level set of A.

Obviously, under the hypothesis of Definition 3.7., the level subsets $A_{\langle\alpha,\beta\rangle}$ are subgroups of G. Indeed, if $x,y\in A_{\langle\alpha,\beta\rangle}$, then we have $A(x)\geq\langle\alpha,\beta\rangle$ and $A(y)\geq\langle\alpha,\beta\rangle$, from which we get

$$A(x.y^{-1})\geq A(x)\wedge A(y^{-1})=A(x)\wedge A(y)\geq\langle\alpha,\beta\rangle\wedge\langle\alpha,\beta\rangle=\langle\alpha,\beta\rangle.$$

Now we shall extend the concept of IFG in G with respect to an IFN using the approach of Anthony-Sherwood [1,2]. For this purpose let $\langle T,S\rangle$ be an IFN in $[0,1]\times[0,1]$, and $\langle x_1,y_1\rangle,\langle x_2,y_2\rangle$ be two intuitionistic fuzzy pairs. Then we define

$$\langle T,S\rangle\ (\langle x_1,y_1\rangle,\langle x_2,y_2\rangle) = \langle\ T\langle x_1,x_2\rangle,\ S\langle y_1,y_2\rangle\ \rangle.$$

Definition 3.8. [cf. 1,2] Let (G,.) be a groupoid. An intuitionistic fuzzy subgroupoid in G with respect to an IFN $\langle T,S\rangle$ is an IFS $A=\langle g,\mu_A,\gamma_A\rangle$ satisfying the condition

$$A(x.y)\ \geq\ \langle T,S\rangle\ (A(x),A(y))\quad \text{for all } x,y\in G.$$

Remarks 3.9. (1) The condition $A(x.y) \geq \langle T,S\rangle\ (\ A(x),\ A(y)\ )$ means exactly that

$$\mu_A(x.y)\geq T(\mu_A(x),\mu_A(y))\quad \text{and}\quad \gamma_A(x.y)\leq S(\gamma_A(x),\gamma_A(y)),$$

for all $x,y\in G$.

(2) If A is an intuitionistic fuzzy subgroupoid in G with respect to an IFN $\langle T,S\rangle$, then μ_A and $1-\gamma_A$ are fuzzy subgroupoids in G with respect to the fuzzy norm T in the sense of Anthony-Sherwood [1,2]. Indeed, since $T(1-\mu,1-\gamma)+S(\mu,\gamma)\leq 1$ for each intuitionistic fuzzy pair $\langle\mu,\gamma\rangle$, we get

$$1-\gamma_A(x.y)\geq 1-S(\gamma_A(x),\gamma_A(y))\geq T(1-\gamma_A(x),1-\gamma_A(y)).$$

Definition 3.10. [cf. 1,2] Let (G,.) be a group. An intuitionistic fuzzy subgroup in G with respect to an IFN $\langle T,S\rangle$ is an intuitionistic fuzzy subgoupoid $A=\langle g,\mu_A,\gamma_A\rangle$ in G with respect to the IFN $\langle T,S\rangle$ satisfying the condition $A(x^{-1})\geq A(x)$ for all $x\in G$.

Proposition 3.11. Let A be an IFG in G with respect to the IFN $\langle T,S\rangle$. Then

(a) $[]A$ is an IFG in G with respect to the IFN $\langle T,T^*\rangle$, where

$T^*(x,y)=1-T(1-x.1-y)$.

(b) $\langle\rangle A$ is an IFG in G with respect to the IFN $\langle S^*,S\rangle$, where

$S^*(x,y)=1-S(1-x.1-y)$.

4. Products of intuitionistic fuzzy subgroups

Let X_1, X_2 be two nonempty sets and let A_1, A_2 be two IFS's in X_1 and X_2, respectively. Then one can define several products IFS's in $X_1\times X_2$. One of these products is given as follows:

$$A_1\times A_2=\langle (x_1,x_2),\ \mu_{A_1}(x_1)\wedge\mu_{A_2}(x_2),\ \gamma_{A_1}(x_1)\vee\gamma_{A_2}(x_2)\ \rangle.$$

Proposition 4.1. If A_1 and A_2 are intuitionistic fuzzy subgroups in G_1 and G_2, respectively, then $G_1\times G_2$ is an intuitionistic fuzzy subgroup in the product group $G_1\times G_2$.

Now we want to give a Anthony-Sherwood version of the previous product. For this purpose on mind, we first give:

Lemma 4.2. For any IFN <T,S> we have

(a) $T(T(x,y),T(z,t)) = T(T(x,z),T(y,t))$ and

(b) $S(S(x,y),S(z,t)) = S(S(x,z),S(y,t))$

for any $x,y,z,t\in[0,1]$.

Let A_1, A_2 be two IFS's in X_1 and X_2, respectively, and let <T,S> be an IFN in $[0,1]x[0,1]$. Then we can define the product IFS A_1xA_2 in X_1xX_2 as follows [cf. 13] :

$$A_1xA_2=<(x_1,x_2),\ T(\mu_{A_1}(x_1),\mu_{A_2}(x_2)),\ S(\gamma_{A_1}(x_1),\gamma_{A_2}(x_2))>.$$

Proposition 4.3. Let G_1, G_2 be two groups, and let A_1, A_2 be two intuitionistic fuzzy subgroups in G_1 and G_2, respectively, with respect to an intuitionistic fuzzy norm <T,S>. Then A_1xA_2 is an intuitionistic fuzzy subgroup in G_1xG_2 with respect to <T,S>.

5. Homomorphic preimages and images of IFG's

Proposition 5.1. Let $f : G \to H$ be a homomorphism between the groupoids G and H. If B is an intuitionistic fuzzy subgroupoid in H, then $f^{-1}(B)$ is an intuitionistic fuzzy subgroupoid in G, too.

Proof. Since $f^{-1}(B)=<g,f^{-1}(\mu_B),f^{-1}(\gamma_B)>$, where $B=<h,\mu_B,\gamma_B>$, and

$$f^{-1}(\mu_B)(x.y)=\mu_B(f(x.y))=\mu_B(f(x).f(y))\geq\mu_B(f(x))\wedge\mu_B(f(y))$$
$$=f^{-1}(\mu_B)(x)\wedge f^{-1}(\mu_B)(y),$$
$$f^{-1}(\gamma_B)(x.y)=\gamma_B(f(x.y))=\gamma_B(f(x).f(y))\leq\gamma_B(f(x))\vee\gamma_B(f(y))$$
$$=f^{-1}(\gamma_B)(x)\vee f^{-1}(\gamma_B)(y),$$

the result follows immediately. ■

In this section we shall also investigate the homomorphic images of intuitionistic fuzzy subgroupoids and intuitionistic fuzzy subgroups. The first result we obtained is similar to that of Eroglu [9]:

Proposition 5.2. Let $f : G \to H$ be a homomorphism between the groupoids G and H. If A is an intuitionistic fuzzy subgroupoid in G, then f(A) is an intuitionistic fuzzy subgroupoid in H, too.

Proof. Let $y_1,y_2\in H$, and $A=<g,\mu_A,\gamma_A>$ be an intuitionistic fuzzy subgroupoid in G. Using a similar technique to that of Eroglu [9], we may easily deduce the inequality

$$\mu_{f(A)}(y_1.y_2)=f(\mu_A)(y_1.y_2)\geq f(\mu_A)(y_1)\wedge f(\mu_A)(y_2)=\mu_{f(A)}(y_1)\wedge\mu_{f(A)}(y_2).$$

On the other hand, we may equally write down

$$f(1-\gamma_A)(y_1 \cdot y_2) \geq f(1-\gamma_A)(y_1) \wedge f(1-\gamma_A)(y_2)$$

$$\Longrightarrow [1-f(1-\gamma_A)](y_1 \cdot y_2) \leq [1-f(1-\gamma_A)](y_1) \vee [1-f(1-\gamma_A)](y_2)$$

$$\Longrightarrow f_-(\gamma_A)(y_1 \cdot y_2) \leq f_-(\gamma_A)(y_1) \vee f_-(\gamma_A)(y_2)$$

$$\Longrightarrow \gamma_{f(A)}(y_1 \cdot y_2) \leq \gamma_{f(A)}(y_1) \vee \gamma_{f(A)}(y_2) .$$

Hence the required result $f(A)(y_1 \cdot y_2) \geq f(A)(y_1) \wedge f(A)(y_2)$ follows immediately. ■

Secondly, we give a generalized version of previous proposition in terms of a continuous intuitionistic fuzzy norm <T,S> as follows:

Proposition 5.3. Let $f : G \rightarrow H$ be a surjective homomorphism between the groupoids G and H. If A is an intuitionistic fuzzy subgroupoid in G with respect to a continuous intuitionistic fuzzy norm <T,S>, then f(A) is an intuitionistic fuzzy subgroupoid in H with respect to <T,S>, too.

Proof. Obvious. ■

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AMS Subject Classification: 04A99.