S-fuzzy subgroup

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Abstract: In this paper, concept of s-fuzzy subgroup of a group G is given, s-fuzzy subgroup is based on s-norm and is a new kind of fuzzy subgroup, we give concepts of normal s-fuzzy subgroup and the lower image of homomorphism and discuss their properties.

Keywords: Group, s-fuzzy subgroup, homomorphism

The concept of fuzzy subgroup of a group G was introduced by Rosenfeld[1] in 1971. Later, J. M. Anthony and H. sherwood generalize the fuzzy subgroup to a t-norm[2]. The fuzzy subgroup in [2] is called a t-fuzzy subgroup by us. A fuzzy subset H of a group G is said to be a t-fuzzy subgroup if t is a t-norm and for all $x, y \in G$

(i) $H(xy) \geqslant t(H(x), H(y))$; (ii) $H(x^{-1}) \geqslant H(x)$

In this paper, we extend this concept to s-norm.

Definition 1. A mapping $S: [0,1] \times [0,1] \to [0,1]$ is called a s-norm if for all $\lambda, \mu, \xi, \eta \in [0,1]$

(1) $S(\lambda, 0) = \lambda, S(\lambda, 1) = 1$;

(2) $S(\lambda, \mu) = S(\mu, \lambda)$;

(3) $\lambda \leqslant \xi, \mu \leqslant \eta \Rightarrow S(\lambda, \mu) \leqslant S(\xi, \eta)$;

(4) $S(\lambda, S(\mu, \xi)) = S(S(\lambda, \mu), \xi)$.

For example $S_0(\lambda, \mu) = \max\{\lambda, \mu\}$ is a s-norm.

Definition 2. Let S_1, S_2 be s-norms. If $S_1(\xi, \eta) \geqslant S_2(\xi, \eta)$ for any $\xi, \eta \in [0,1]$, then we denote $S_1 \geqslant S_2$.

Remark. (1) Let S be a s-norm, then $S \geqslant S_0$.

(2) $S = S_0$ if and only if $S(\xi, \xi) = \xi, \forall \xi \in [0,1]$.

Definition 3. Let S be a s-norm. A fuzzy subset H of a group G is said to be a s-fuzzy subgroup if for all $x, y \in G$

(1) $H(xy) \leqslant S(H(x), H(y))$; (2) $H(x^{-1}) \leqslant H(x)$.

Theorem 1. Let H be a fuzzy subset of a group G, e is identity of group G. If

$$S(H(x), H(e)) \leqslant H(x), \forall x \in G$$

then H is a s-fuzzy subgroup of group G if and only if $H(xy^{-1}) \leqslant S(H(x), H(y))$.

Proof. "$\Rightarrow$" Let H is a s-fuzzy subgroup of G then

$H(xy^{-1}) \leqslant S(H(x), H(y^{-1})) \leqslant S(H(x), H(y))$.

"$\Rightarrow$" $H(xy^{-1}) \leqslant S(H(x), H(y)), \forall x, y \in G$.

Let $x = e$, then $H(y^{-1}) \leqslant S(H(e), H(y)) \leqslant H(y)$, it follows that $H(xy) = H(x(y^{-1})^{-1})$

$\leqslant S(H(x), H(y^{-1})) \leqslant S(H(x), H(y))$.

So H is a s-fuzzy subgroup of group G.

Definition 4. (1) Let G and G' be groups. $f: G \to G'$ is a homomorphism, H is a fuzzy subset of G. We define:

$$f(H)(y) \triangleq \inf_{f(x)=y} H(x)$$

then $f(H)$ is said to be lowerimage of H.

(2)f is said to be accessible if for all $y \in f(G)$, there exists $x_0 \in G$ such that $f(x_0)=y$ and $f(H(y))=H(x_0)$.

Theorem 2. Let G and G' be groups. $f: G \to G'$ is a group homomorphism.

(1)If H is a s-fuzzy subgroup of G and f is accessible, then $f(H)$ is a s-fuzzy subgroup of G'.

(2)Let B be a s-fuzzy subgroup of G', then $f^{-1}(B)$is a s-fuzzy subgroup of G. where $f^{-1}(B)(x)=B(f(x))$.

Proof(1) Let H be a s-fuzzy subgroup of G, for any $y_1, y_2 \in G'$, Let $f(x_i^0)=y_i$ and $f(H)(y_i)=\inf_{f(x)=y_i} H(x)=H(x_i)(i=1,2)$, then

$$f(H)(y_1y_2)=\inf_{f(x)=y_1y_2} H(x) \leqslant \inf_{\substack{f(x_1)=y_1\\ f(x_2)=y_2}} H(x_1x_2)$$

$$\leqslant \inf_{\substack{f(x_1)=y_1\\ f(x_2=y_2}} (S(H(x_1), H(x_2)) \leqslant S(H(x_1^0), (H(x_2^0)) = S(f(H_1)(y_1), f(H)(y_2))$$

$f(H)(y_1^{-1}) = \inf_{f(x_1^{-1})=y_1^{-1}} H(x_1^{-1}) = \inf_{f(x_1)=y_1)} H(x_1) = f(H)(y_1)$. So $f(H)$ is a s-fuzzy subgroup of G'.

(2)is clear.

Definition 5. Let H be a s-fuzzy subgroup of group G, for any $a \in G$ we define

$$(aH)(x)=H(a^{-1}x), \forall x \in G;$$
$$(Ha)(x)=H(xa^{-1}), \forall x \in G.$$

then aH and Ha are called as left coset and right coset of H respectively.

Theorem 3. Let H be a s-fuzzy subgroup of group G, if for any $x \in G$ we have

$$S(H(x), H(e)) \leqslant H(x),$$

then $aH=bH$ if and only if $H(a^{-1}b)=H(e)$.

Definition 6. A s-fuzzy subgroup H of group G is said to be a normal s-fuzzy subgroup of G if $H(xyx^{-1}) \leqslant H(y), \forall$ x,y$\in$G.

Theorem 4. Following conditions are equivalent

(1)H is a normal s-fuzzy subgroup of group G;

(2)$H(xy)=H(yx), \forall x, y \in G$;

(3)$aH=Ha, \forall a \in G$

Theorem 5. Let G and G' be groups $f: G \to G'$ is an epimorphism

(1)If H is a normal s-fuzzy subgroup of G, Then $f(H)$ is a normal s-fuzzy subgroup of G'.

(2)If B is a normal s-fuzzy subgroup of G' ,then $f^{-1}(B)$ is a normal s-fuzzy subgroup of G

Proof:By theorem 2. we know that $f(H)$ and $f^{-1}(B)$ and s-fuzzy subgroup.

(1)For any $y_1, y \in G'$

$$f(H)(y_1 y_1^{-1}) = \inf_{f(x)=y_1 y y_1^{-1}} H(x) \leqslant \inf_{\substack{f(x)=y \\ f(x_1)=y_1}} H(x_1 x x_1^{-1}) \leqslant \inf_{f(x)=y} H(x) = f(H)(y).$$

(2) For any $x, y \in G$

$$f^{-1}(B)(xyx^{-1}) = B(f(xyx^{-1})) = B(f(x)f(y)f(x)^{-1}) \leqslant B(f(y)) = f^{-1}(B)(y).$$

Definition 7. Let A, B be fuzzy subsets of group G, A fuzzy subset AB of G of the form $(AB)(x) = \inf_{a\in G} S(A(a), B(a^{-1}x))$ is said to be product of A and B.

Theorem 6. Let H be a normal fuzzy subgroup of group G and for any $x \in G$ $S(H(x), H(e)) \leqslant H(x)$,then $(xH)(yH) = (xy)H$.

Proof. For any $z \in G$

$$(xyH)(z) = H((xy)^{-1}z) = H(y^{-1}x^{-1}z) = H(y^{-1}x^{-1}ayy^{-1}a^{-1}z)$$
$$\leqslant S(H(y^{-1}x^{-1}ay), H(y^{-1}a^{-1}z)) = S((yHy^{-1})(x^{-1}a), H(y^{-1}a^{-1}z))$$
$$\leqslant S(H(x^{-1}a), H(y^{-1}a^{-1})z)) = S((xH)(a), (yH)(a^{-1}z))$$

So $(xyH)(z) \leqslant \inf_{a\in G} S((xH)(a), (yH)(a^{-1}z)) = ((xH)(yH))(z)$

Hence $(xy)H \subseteq (xH)(yH)$

For any $z \in G$

$$((xH)(yH))(z) = \inf_{a\in G} S((xH)(a), (yH)(a^{-1}z)) \leqslant S((xH)(x), (yH)(x^{-1}z))$$
$$= S(H(e), H(y^{-1}x^{-1}z)) \leqslant H(y^{-1}x^{-1}z) = (xyH)(z)$$

So $(xH)(yH) \subseteq (xy)H$. Hence $(xH)(yH) = (xy)H$.

Theorem 7. Let H be a normal s-fuzzy subgroup of group G and $G/H = \{aH \mid a \in G\}$. Let $(aH)(bH) = abH$,then G/H form a group, G/H is called the quotient group of G module H.

Theorme 8. Let H be a normal s-fuzzy subgroup of group G, Let $N = \{x \mid x \in G$ and $H(x) = H(e)\}$ then N is a normal subgroup of G and G/H is isomorphic with G/N.

References

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