Review of the book

"Fuzzy Set Theory — Basic Concepts, Techniques and Bibliography"

by Robert LOWEN

Kluwer Academic Publishers, Dordrecht, 1996, 408 p.

Fuzzy sets are thirty-two years old, and a great number of books (collections of articles and to a less extent, monographs) have been published about them, especially during the last ten years. The book written by Robert Lowen (one of the pioneers of the introduction of fuzzy sets into mathematics, especially topology) is a monograph whose intended aim is to provide an application-oriented reader with an elementary presentation, but still mathematically rigorous, of the basic concepts of fuzzy set theory. Let us briefly present the contents of the book. Its expository part is made of six chapters.

Chapter 1 gives a background on elementary set theory (set-operations, functions and relations) and partially ordered sets (with emphasis on lattice structures). Chapter 2 introduces fuzzy sets (with [0,1]-valued membership functions), starting with mathematical examples, then pointing out the underlying completely distributive complete lattice, and Brouwerian lattice, structures. It finally defines the image of a fuzzy set by a function, and more generally, the extension principle and the composition of fuzzy relations. The chapter ends with the α-level cuts representation of a fuzzy set (in terms of a family of ordinary sets). Chapter 3, after introducing fuzzy complements, provides an extensive study of fuzzy set operations pointwisely defined on membership degrees by means of associative, symmetric, monotonic two-place operations called t-norms (for intersections) and t-conorms (for unions). A background on t-norms and t-conorms (short for triangular norms and conorms since these operations were first encountered in stochastic geometry in the expression of triangular inequalities) is given, including their additive and multiplicative representations when they exist. An important part of this long chapter (more than eighty pages) is used for an extensive presentation of all the examples in the fuzzy set literature of t-norms and t-conorms including in each case a graphical representation. Parametrized families covering the three basic t-norms (min, product, and the "bounded product" $\max(0, x + y - 1)$) as well as other families are also surveyed. The chapter ends with results on DeMorgan triples, i.e., a triple of t-norm, t-conorm and negation functions fulfilling DeMorgan laws. The ten pages of Chapter 4, devoted to special types of fuzzy sets, provide more definitions and results concerning normal fuzzy sets, convex fuzzy sets (defined only on a vectorspace), and piecewise linear fuzzy sets and compact fuzzy sets on the real line. Chapter 5 deals with fuzzy real numbers. Two different views are considered and presented. In the first case, called probabilistic view, fuzzy real numbers constitute stochastic quantities, and are non-decreasing functions from 0 to 1 on the real line, understood as (cumulative) probability distribution functions. In this view the real number x is represented by the step function which is the characteristic function of the interval $[x, +\infty)$. In the second view, fuzzy real numbers rather generalize the notion of real intervals. The extension principle, in both cases, enables the addition of fuzzy real numbers to be defined. Lagrange interpolation theorem is then generalized to a fuzzy real numbers-valued function. Chapter 6, entitled "Fuzzy Logic", deals with fuzzy connectives and generalizations of the syllogism and modus ponens inference rules. First, the truth-table view of binary connectives in classical logic is recalled, as well as the laws which hold in Boolean algebra. Lattice-based extensions of binary connectives to fuzzy connectives are presented. More general implication connectives of the form $(1 - x) \perp y$ (where $\perp$ is a t-conorm and x and y belong to $[0,1]$), are then studied (with the related connectives). A "probabilistic" interpretation of such connectives is also given (where a disjunctive operator $D(x,y)$ is associated to a conjunction operator $C(x,y)$ through the relation $D(x,y) = x + y - C(x,y)$, rather than through the relation $D(x,y) = 1 - C(1 - x, 1 - y)$ as in the standard t-(co)norm view. The implication connective $x \rightarrow y$ is generated in both cases as "not x or y". Implication connectives defined through residuation ($x \rightarrow y = \sup\{t \mid t \in [0,1], x * t \leq y\}$ where $*$ is a conjunction connective) are not considered. Necessary and sufficient conditions on x and y for having the following counterpart of modus ponens in terms of degrees of truth

$$x * [(1 - x) \perp y] \leq y,$$

where $*$ (resp. $\perp$) is a conjunction (resp. disjunction) connective are given for many possible choices of $*$ and $\perp$. Each of these six chapters is completed at their end by several pages of bibliographical notes. They point to selected references where the reader can find more results which further develop the issues raised in the chapter. Apart from a subject index, the last part of the book (about 170 pages) is made of two alphabetical lists, one for books devoted to fuzzy sets or related topics, the other obviously much longer, collects approximately 1900 research articles, which provides the book with a rich bibliographical apparatus.

This monograph, as an introduction to fuzzy sets, is rather original and unique in the sense that it emphasizes the basic mathematical constructs underlying fuzzy sets, which is not often done is such a systematic way. However it should be clear that this volume only focuses on basic concepts, and that some topics, which might be expected in a more standard introductory volume, are only briefly mentioned, such as fuzzy similarity and fuzzy orderings relations, fuzzy relation equations, or simply omitted such as approximate reasoning in the sense of Zadeh for instance (indeed this topic is more application-oriented). Moreover this book deals only with fundamentals of fuzzy sets *stricto sensu*, since neither possibility theory originated by Zadeh, nor Sugeno's fuzzy integrals, which are basic contributions to a non-probabilistic modelling of uncertainty, are considered in this volume. These limitations are acknowledged in the introduction by the author himself.

Didier DUBOIS & Henri PRADE