The convergence of fuzzy complex valued series

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Abstract In this paper, on the basis of the fuzzy complex numbers introduced by J.J. Buckley [2], by means of the definition of the sequence of fuzzy complex numbers, we give the concepts of the fuzzy complex valued series and convergence (divergence). And then, we study the convergence criterions and its some elementary properties.

Keywords Fuzzy number; fuzzy complex number; fuzzy complex valued series; convergence; support:

1 Introduction

Since the concept of fuzzy complex numbers was given at first by J.J. Buckley [1] in 1987, a lot of scholars had studied the theory of fuzzy complex numbers widely and deeply, and established a rudiamental frame of fuzzy complex analysis. For example, the differentiability and integrability of fuzzy complex valued functions on complex plane K had been discussed constantly in [3, 4]. In the meantime its contour integral was being established. On the basis of [2], Zhang[5] discussed the limit theory of the sequence of fuzzy complex numbers in detail, giving a series of results about limit theory of being similar to real numbers in classical mathematics analysis.

On an ordinary fuzzy set space, the fuzzy series was defined and studied in [6]. The faults of the definition of [6] were pointed out and revised by [7].

In this paper, on the basis of the sequence of fuzzy complex numbers introduced by Zhang [5], by introducing the concept of the support of fuzzy complex numbers, we give the convergent (divergent) definitions of fuzzy complex valued series. Consequently we obtain the necessary and sufficient conditions on convergence of fuzzy complex valued series, comparison criterion and some elementary properties on convergence.

2 Fuzzy numbers and fuzzy complex numbers.

In this section, as a preparation, we first remind some concepts and results on fuzzy numbers and fuzzy complex numbers. Here R is the set of all real numbers.

Definition 2.1 Let A be a fuzzy set on R. If A is both normal and fuzzy convex, then A is called a normal fuzzy set on R.

Definition 2.2 Let A be a normal fuzzy set on R. If A is upper semi-continuous, and the closure cl(supp$A$) of its support(supp$A = \{x \in R \mid A(x) > 0\}$) is compact, then A is called a fuzzy number on R.

Let $F(R)$ denote the set of all fuzzy numbers on R . From definition 2.2, for arbitrary $A \in F(R)$ and $\lambda \in (0,1]$, we know easily that the cut - set $A_\lambda = \{x \in R \mid A(x) \geqslant \lambda\}$ is a closed bounded interval on R . Thus, we may let $A_\lambda = [A_\lambda^-, A_\lambda^+]$ where $A_\lambda^- \leqslant A_\lambda^+, A_\lambda^-, A_\lambda^+ \in R, \lambda \in (0,1]$

Let $A, B \in F(R)$, on extended addition (subtraction) of fuzzy numbers and multiplication between a number and a fuzzy number, we define the following operations:

(1) $(A \pm B)(x) = \bigvee_{y \pm z = x} (A(y) \wedge B(z))$, where $x, y, z \in R$;

(2) $(kA)(x) = \begin{cases} A(\frac{x}{k}) & k \neq 0 \\ 0 & x \neq 0, k = 0 \\ 1 & x = 0, k = 0 \end{cases}$, where $k, x \in R$

Define the order of fuzzy numbers as follows:

(3) $A \leqslant B$ iff $A_\lambda^- \leqslant B_\lambda^-, A_\lambda^+ \leqslant B_\lambda^+$, for each $\lambda \in (0,1]$;

(4) $A = B$ iff $A_\lambda^- = B_\lambda^-, A_\lambda^+ = B_\lambda^+$, for each $\lambda \in (0.1]$

(5) $A < B$ iff $A \leqslant B$and $A \neq B$

Let the sequence of fuzzy numbers $\{A_n\} \subset F(R)$, $n = 1,2\cdots$. Define the membership function of its infinite sum as

$$(\sum_{n=1}^{\infty} A_n)(x) = \sup\{\bigwedge_{n=1}^{\infty} A_n(x_n) \mid x = \sum_{n=1}^{\infty} x_n, \sum_{n=1}^{\infty} \mid x_n \mid < +\infty\}$$

Definition 2.3 Let $A, B \in F(R)$, K be a set of all complex numbers we define

$$(A,B): K \to [0,1]$$

$$z = x + yi \to A(x) \wedge B(y), \text{where } x, y \in R$$

Then (A, B) is called a fuzzy complex number on K . A and B is said to a real part and an imaginary part of (A, B) respectively. We write $Z = (A, B), A = ReZ, B = ImZ$, or $Z = (ReZ, ImZ)$.

Especially, whenever $B = ImZ = \bar{0}$ ($\bar{0} \in F(R)$) , we define

$(A, \bar{0}) = A$, i.e., a fuzzy complex number is a generalization of a fuzzy number.

Let $F(K)$ denote the set of all fuzzy complex numbers on K , for every $c \in K$, writting $c = a + bi = (a, b)$, where $a, b \in R$. we define

$$c(z) = \begin{cases} 1 & x = a, y = b \\ 0 & \text{otherwise} \end{cases}, \quad \text{whenever } z = (x, y) \in K.$$

Obviously, $c \in F(K)$, i.e., a fuzzy complex number is also a generalization of an ordinary complex number.

About extended addition (subtraction) of fuzzy complex numbers and multiplication between a real number and a fuzzy complex number, we define the operations as follows:

(1) $Z_1 \pm Z_2 = (ReZ_1 \pm ReZ_2, ImZ_1 \pm ImZ_2)$

(2) $kZ_1 = (kReZ_1, kImZ_1)$ and

$Re(Z_1 \pm Z_2) = ReZ_1 \pm ReZ_2$; $Im(Z_1 \pm Z_2) = ImZ_1 \pm ImZ_2$;

$Re(kZ_1) = kReZ_1$; $Im(kZ_1) = kImZ_1$

On the order of fuzzy complex numbers, we make the definition as follows:

③ $Z_1 \leqslant Z_2$ iff $ReZ_1 \leqslant ReZ_2$, $ImZ_1 \leqslant ImZ_2$;

④ $Z_1 = Z_2$ iff $ReZ_1 = ReZ_2$, $ImZ_1 = ImZ_2$;

⑤ $Z_1 < Z_2$ iff $Z_1 \leqslant Z_2$ and $Z_1 \neq Z_2$.

Definition 2.4 Let the sequence of fuzzy complex numbers $\{Z_n\} \subset F(K)$, $n = 1, 2\cdots$. Then $Z_1 + Z_2 + \cdots + Z_n + \cdots$ is called a fuzzy complex valued series. It is simply denoted as $\sum_{n=1}^{\infty} Z_n$. We define its infinite sum and its membership function respectively as follws:

$$\sum_{n=1}^{\infty} Z_n = \left(\sum_{n=1}^{\infty} ReZ_n, \sum_{n=1}^{\infty} ImZ_n\right);$$

$$\left(\sum_{n=1}^{\infty} Z_n\right)(z) = \left(\sum_{n=1}^{\infty} ReZ_n\right)(x) \wedge \left(\sum_{n=1}^{\infty} ImZ_n\right)(y) \qquad \text{where } z = x + yi,\ x, y \in$$

R. and Z_n is called a general term of $\sum_{n=1}^{\infty} Z_n$.

3 Convergence criterions

In this section, we get over the faults of definition in [6], by introducing the concept of support of fuzzy complex numbers, define and discuss the convergence and divergence of fuzzy complex valued series.

Defintion 3.1 Let $Z \in F(K)$, Supp$Z = \{z = x + yi \in K \mid (ReZ)(x) > 0, (ImZ)(y) > 0, x, y \in R\}$. Then supp$Z$ is called the support of fuzzy complex number Z

Definition 3.2 Let the sequence of fuzzy complex numbers $\{Z_n\} \subset F(K)$, $n = 1, 2, \cdots$. If for arbitrary $z_n = x_n + y_n i \in \text{supp}Z_n$, the relevant number - term series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are all convergent. Then we call fuzzy complex valued series $\sum_{n=1}^{\infty} Z_n$ convergence. Otherwise, if there exists at least a sequence of complex numbers $\{z_n\} \subset \text{supp}Z_n$ such that the number - term series $\sum_{n=1}^{\infty} x_n$ or $\sum_{n=1}^{\infty} y_n$ diverge. Then we call $\sum_{n=1}^{\infty} Z_n$ divergence.

Theorem 3.1 Let $\sum_{n=1}^{\infty} Z_n$ be a fuzzy complex valued series. then $\sum_{n=1}^{\infty} Z_n$ is convergent iff for any sequences $\{x_n\} \subset \text{supp}(ReZ_n)$, $\{y_n\} \subset \text{supp}(ImZ_n)$, where $n = 1, 2, \cdots$, the number - term series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are all convergent.

Proof. Making use of definition 3.1 and definition 3.2, it is clear. So omit it.

Theorem 3.2 (Comparison criterion). Let $\sum_{n=1}^{\infty} Z_n$, $\sum_{n=1}^{\infty} Z_n'$ be two fuzzy complex valued series respectively, and they satisfy $Z_n \leqslant Z_n'$, $n = 1, 2, \cdots$. Then

(1) If $\sum_{n=1}^{\infty} Z_n'$ converges, then $\sum_{n=1}^{\infty} Z_n$ is also convergent;

(2) If $\sum_{n=1}^{\infty} Z_n$ diverges, then $\sum_{n=1}^{\infty} Z_n'$ is also divergent.

Proof. (1) For any $z_n = x_n + y_n i \in \text{supp} Z_n$, $n = 1, 2, \cdots$, i.e.,
$(ReZ_n)(x_n) > 0$, $(ImZ_n)(y_n) > 0$

By known conditions and the definition of the order of fuzzy complex numbers, we obtain $(ReZ_n')(x_n) \geqslant (ReZ_n)(x_n) > 0$, $(ImZ_n')(y_n) \geqslant (ImZ_n)(y_n) > 0$.

Therefore $\{x_n\} \subset \text{supp}\,(ReZ_n')$, $\{y_n\} \subset \text{supp}(ImZ_n')$. From the convergent definition of $\sum_{n=1}^{\infty} Z_n'$, we know that $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are all convergent.

Thus by theorem 3.1, we get that $\sum_{n=1}^{\infty} Z_n$ is convergent.

(2) Otherwise, if $\sum_{n=1}^{\infty} Z_n'$ converges, take advantage of (1), we can derive from immediately that $\sum_{n=1}^{\infty} Z_n$ is convergent. But this contradicts the known condition.

And so $\sum_{n=1}^{\infty} Z_n'$ is divergent.

4 Convergence

In this section, we further study the properties of convergence of fuzzy complex valued series. Consequently we obtain some results being similar to classical mathematics analysis.

Theorem 4.1 If fuzzy complex valued series $\sum_{n=1}^{\infty} Z_n$ converges, and $0 \neq k \in R$. Then $\sum_{n=1}^{\infty} (kZ_n)$ is also convergent, and $\sum_{n=1}^{\infty} (kZ_n) = k \sum_{n=1}^{\infty} Z_n$.

Proof. First, for arbitrary $x_n + y_n i \in \text{supp}(kZ_n)$, $n = 1, 2, \cdots$, we have
$(kReZ_n)(x_n) = (Re(kZ_n))(x_n) > 0$ and

$(kImZ_n)(y_n) = (Im(kZ_n))(y_n) > 0,$

In the light of multiplication operation between a real number and a fuzzy number, whenever $k \neq 0$, we have

$(ReZ_n)(\frac{x_n}{k}) > 0, (ImZ_n)(\frac{y_n}{k}) > 0$, i.e.,

$\frac{x_n}{k} \in \text{supp}ReZ_n, \frac{y_n}{k} \in \text{supp}ImZ_n$

By the convergence of $\sum_{n=1}^{\infty} Z_n$ and theorem 3.1, we can derive from that $\sum_{n=1}^{\infty} \frac{x_n}{k}$ and $\sum_{n=1}^{\infty} \frac{y_n}{k}$ are all convergent, i.e., $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are also convergent.

Hence $\sum_{n=1}^{\infty} kZ_n$ is convergent.

Second. For each $x \in R$ and $k \neq 0$, by the definition of infinite sum, we have

$$\begin{aligned}
(\sum_{n=1}^{\infty} k(ReZ_n))(x) &= \sup\{\bigwedge_{n=1}^{\infty}(kReZ_n)(x_n) \mid x = \sum_{n=1}^{\infty} x_n, \sum_{n=1}^{\infty} \mid x_n \mid < +\infty\} \\
&= \sup\{\bigwedge_{n=1}^{\infty}(ReZ_n)(\frac{x_n}{k}) \mid \frac{x}{k} = \sum_{n=1}^{\infty} \frac{x_n}{k}, \sum_{n=1}^{\infty} \mid \frac{x_n}{k} \mid < +\infty\} \\
&= (\sum_{n=1}^{\infty} ReZ_n)(\frac{x}{k}) \\
&= k(\sum_{n=1}^{\infty} ReZ_n)(x).
\end{aligned}$$

Therefore $\sum_{n=1}^{\infty} k(ReZ_n) = k(\sum_{n=1}^{\infty} ReZ_n)$

Similarly, we can obtain $\sum_{n=1}^{\infty} k(ImZ_n) = k(\sum_{n=1}^{\infty} ImZ_n)$.

consequently, we have

$$\begin{aligned}
\sum_{n=1}^{\infty}(kZ_n) &= (\sum_{n=1}^{\infty} Re(kZ_n), \sum_{n=1}^{\infty} Im(kZ_n)) \\
&= (k\sum_{n=1}^{\infty} ReZ_n, k\sum_{n=1}^{\infty} ImZ_n) \\
&= k\sum_{n=1}^{\infty} Z_n.
\end{aligned}$$

Lemma 4.1 If fuzzy valued series $\sum_{n=1}^{\infty} A_n$ and $\sum_{n=1}^{\infty} B_n$ are all convergent, $\{A_n, B_n\} \subset F(R)$, $n = 1, 2, \cdots,$. Then $\sum_{n=1}^{\infty}(A_n \pm B_n)$ is also convergent, and $\sum_{n=1}^{\infty}(A_n \pm B_n) = \sum_{n=1}^{\infty} A_n \pm \sum_{n=1}^{\infty} B_n$.

Proof. Refer to [6].

Theorem 4.2 Let fuzzy complex valued series $\sum_{n=1}^{\infty} Z_n$ and $\sum_{n=1}^{\infty} Z_n'$ be all convergent. Then $\sum_{n=1}^{\infty}(Z_n \pm Z_n')$ is also convergent, and $\sum_{n=1}^{\infty}(Z_n \pm Z_n') = \sum_{n=1}^{\infty} Z_n \pm \sum_{n=1}^{\infty} Z_n'$.

Proof. Without loss of generality, we choose arbitrary $x_n + y_n i \in \text{supp}(Z_n + Z_n')$, $n = 1, 2, \cdots$. Then $(ReZ_n + ReZ_n')(x_n) = (Re(Z_n + Z_n'))(x_n) > 0$, and

$$(ImZ_n + ImZ_n')(y_n) = (Im(Z_n + Z_n'))(y_n) > 0.$$

From the extended addition operation of fuzzy numbers, we know that there exists sequences of numbers $\{x_n'\}, \{y_n'\} \subset R$ and $\{x_n''\}, \{y_n''\} \subset R$ such that $x_n' + y_n' = x_n$, $x_n'' + y_n'' = y_n$, $n = 1, 2, \cdots$, and

$$(ReZ_n)(x_n') \wedge (ReZ_n')(y_n') > 0,$$

$$(ImZ_n)(x_n'') \wedge (Im(Z_n'))(y_n'') > 0$$

Hence $\quad x'_n \in \text{supp}(\text{Re}Z_n),\ y'_n \in \text{supp}(\text{Re}Z'_n),$

$$x''_n \in \text{supp}(\text{Im}Z_n),\ y''_n \in \text{supp}(\text{Im}Z'_n),$$

Utilizing theorem 3.1, we know that $\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} x_n' + \sum_{n=1}^{\infty} y_n'$ and

$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} x_n'' + \sum_{n=1}^{\infty} y_n''$ are all convergent.

Consequently, by definition 3.2,

We obtain that fuzzy complex valued series $\sum_{n=1}^{\infty}(Z_n + Z_n')$ is convergent.

Additionally, by theorem 4.1, letting k = −1, then we can get that $\sum_{n=1}^{\infty}(Z_n - Z_n')$ is also convergent.

Finally, by Lemma 4.1, we have

$$\sum_{n=1}^{\infty}(Z_n \pm Z_n') = \sum_{n=1}^{\infty} Z_n \pm \sum_{n=1}^{\infty} Z_n'$$

Corollary 4.2′ If fuzzy complex valued series $\sum_{n=1}^{\infty} Z_n$ and $\sum_{n=1}^{\infty} Z_n'$ are all convergent. real numbers $k, l \neq 0$. Then $\sum_{n=1}^{\infty}(kZ_n \pm lZ_n')$ is also convergent and $\sum_{n=1}^{\infty}(kZ_n \pm lZ_n') = k\sum_{n=1}^{\infty} Z_n \pm l\sum_{n=1}^{\infty} Z_n'$.

Theorem 4.3 If we add to or cut out arbitrary finite terms of a fuzzy complex valued series, then its convergence (or divergence) is not influenced.

Proof. Without loss of generality, let $\sum_{n=1}^{\infty} Z_n$ converge, where $\{Z_n\} \subset F(K), n \geqslant 1$. Choose an arbitrary fixed natural number $m \in N$, for any $z_n = x_n + y_n i \in \text{supp} Z_n$, whenever $n \geqslant m$.

Evidently, number - term series $\sum_{n=m}^{\infty} x_n$ and $\sum_{n=m}^{\infty} y_n$ are all convergent, of course, adding respectively $m-1$ terms to $\sum_{n=m}^{\infty} x_n$ and $\sum_{n=m}^{\infty} y_n$. Then $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are still convergent. And so the proof is completed.

Theorem 4.4 If a fuzzy complex valued series converges, then the fuzzy complex valued series after adding arbitrary a parenthesis to it is still convergent, and they converge to the same fuzzy complex number.

Proof Omit it.

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