FIXED POINTS FOR FUZZY MAPPINGS DEFINED ON UNBOUNDED SETS IN BANACH SPACES

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ABSTRACT: This paper bring forward the concept of nonexpansive type fuzzy mappings in Bancach spaces, for these mappings defined on Unbounded Sets, we give some fixed point theorems, our theorems improve and generalize the corresponding recent important results.

KEY WORDS AND PHRASES: Fuzzy mathematics, fuzzy analysis, fuzzy mapping, nonexpansive type fuzzy mapping, fixed point.

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1 INTRODUCTION AND PRELIMINARIES

It is known that, for a wide class of unbounded closed convex sets, multi-valued nonexpansive point-compact self-mappings may exist which fail to have a fixed point, so it is importmt to study the problem of existence of fixed points for nonexpansive maps defined on closed convex unbounded subsets. In [1,2,3] analogous problems are treated for single-valued mappings. In 1991, Marino. G, Canetti. A give some fixed point theorems for multivalued mappings defined on Unbounded Sets in Banach spaces [4,5], these theorems unifies and improves the corrspoading results in [1,2,3]. In 1996, Zhang Xian give some new improvements [6]. This paper bring forward the concept of nonexpansive type fuzzy mappings in Banach spaces, for these mappings defined on Unbounded sets, we give some fixed point theorems, our theorems improve and generalize the corresponding results in [1,2,3,4,5,6].

Throughout this paper, let $(X, \|\cdot\|)$ be a Banach space, $K \subseteq X$, $CB(X)(CB(K))$ be collection of all nonempty closed bounded subsets of $X(K)$, $C(X)(C(K))$ be collection of all nonempty compact subsets of $X(K)$, for any $A \in CB(X)$, $\overline{Co}(A)$ be the closed convex hull of A, $d(x,A) = \inf\limits_{y \in A} \| x - y \|$, for any $A, B \in CB(X)$, we note with $H(A,B)$ the Hausdorff distance induced by the norm of X, i. e.

$$H(A,B) = max\{\sup_{a \in A} d(a,B), \sup_{b \in B} d(b,A)\}$$

A multivalued mapping $T: K \to CB(X)$ is said to be Lipschitzian if

$$H(Tx,Ty) \leqslant L \| x - y \|, \forall\, x,y \in K, \tag{1.1}$$

$$Where\ L \geqslant 0$$

T is said to be a contraction if $L < 1$ and nonexpansive if $L = 1$.

For $K \subseteq X$, a mapping $A: K \to [0,1]$ is called a fuzzy subset over K, we denote by $\mathscr{F}(K)$ the family of all fuzzy subsets over K, a mapping $F: K \to \mathscr{F}(K)$ is called fuzzy mapping over K, let $A \in \mathscr{F}(K)$, $\alpha \in [0,1]$, set $A\alpha = \{x \mid A(x) \geqslant \alpha, x \in K\}$ is called the α-cut set of A.

Let $F: K \to \mathscr{F}(K)$, $O(x): K \to [0,1]$, throughout this paper we denote $(Fx)_{O(x)}$ by $\widetilde{F}x$ set, i. e., $\widetilde{F}x = (Fx)_{O(x)} \forall\, x \in K$.

DEFINITION 1. 1. Let $\{F: K \to \mathscr{F}(K)$ be a fuzzy mapping, $O(x)$: $K \to (0,1]$ be a function, if $\forall\, x \in K$, $\widetilde{F}x \in C(K)$ and

$$H(\widetilde{F}x,\widetilde{F}y) \leqslant \| x - y \|, \forall\, x,y \in K \tag{1.2}$$

then the say that F for $O(x)$ be the nonexpansive type fuzzy mappings.

DEFINITION 1. 2. Let $F: K \to \mathscr{F}(K)$ be a fuzzy mapping, if $p \in K$ such that $Fp(p) \geqslant \alpha$, then we say that the fixed degree of p for $F \geqslant \alpha$, if $Fp(p) = \max\limits_{u \in K} Fp(u)$, we say that F has the maximun fixed degree at p, or that p is a fixed point of F.

DEFINITION 1. 3. For $x,y \in X, K \subseteq X$, let $GL(x,y,K) = \{z \in K \mid \| z - y \| < \| z - x \| \}$, for $x \in X, A \subseteq X, K \subseteq X$, let $GL(x,A,K) = \{z \in K \mid \exists\ a \in A, \| z - a \| < z - x \| \} = \bigcup\limits_{a \in A} GL(x,a,K)$.

2 MAIN RESULTS

THEOREM 2. 1. Let X be a Bamach space whose bounded closed convex subsets have the fixed point property for multivalued nonexpansive point-compact self-mappings, K be a closed convex subset of $X, F: K \to \mathscr{F}(K)$ be a fuzzy mapping.

(1) If there exsits a function $O(x): K \to (0,1]$ such that for all $x \in K$, $\widetilde{F}x \in C(K)$, F for $O(x)$ be the nonexpansive type fuzzy mappings, and for some $x_0 \in K$ the set $GL(x_0, \overline{Co}\widetilde{F}x_0, K)$ is bounded, then there exisists $p \in K$ such that $Fp(p) \geqslant O(p)$.

(2) In particular, if F for $O(x) = \max\limits_{u \in K} Fx(u)$ satisfies the conditions in (1), then F has fixed point $p \in K$.

PROOF. If $x_0 \in \widetilde{F}x_0$, thus $Fx_0(x_0) \geqslant O(x_0)$, we have $p = x_0$. Assume $x_0 \notin \widetilde{F}x_0$, set

$$R = 4\sup\{H(\{z\}, \overline{Co}\widetilde{F}x_0), z \in GL(x_0, \overline{Co}\widetilde{F}x_0, K)\}$$

$$S = \{z \in K \mid \exists\ v \in \overline{Co}\widetilde{F}x_0,\ \| y - v \| \leqslant R\}$$

it is easy to prove that S is nonempty, closed, convex and bounded, we will show that $\forall\ x \in S, \widetilde{F}x \subseteq S$.

Since K be a closed convex subset of $X, \widetilde{F}x_0 \subseteq K$, therefore $\overline{Co}\widetilde{F}x_0 \subseteq K$, for any $u \in \overline{Co}\widetilde{F}x_0, \dfrac{x_0 + 2u}{3} \in K$.

If $z \in S \cap GL(x_0, \overline{Co}\widetilde{F}x_0, K)$, by $z \in GL(x_0, \overline{Co}\widetilde{F}x_0, K)$ we have $H(\{z\}, \overline{Co}\widetilde{F}x_0) \leqslant \dfrac{R}{4}$, for any $u \in \overline{Co}\widetilde{F}x_0$, by $\dfrac{x_0 + 2u}{3} \in K$ and $\| \dfrac{x_0 + 2u}{3} - u \| = \dfrac{1}{3} \| x_0 - u \| < \dfrac{2}{3} \| x_0 - u \| = \| \dfrac{x_0 + 2u}{3} - x_0 \|$, we have $\dfrac{x_0 + 2u}{3} \in GL(x_0, u, K) \subseteq GL(x_0, \overline{Co}\widetilde{F}x_0, K)$, thus $\dfrac{R}{4} \geqslant H(\{\dfrac{x_0 + 2u}{3}\}, \overline{Co}\widetilde{F}x_0) \geqslant \| \dfrac{x_0 + 2u}{3} - u \| = \dfrac{1}{3} \| x_0 - u \|$, therefore $\| x_0 - u \| \leqslant \dfrac{3}{4}R$, we have $\| z - x_0 \| \leqslant \| z - u \| + \| u - x_0 \| \leqslant \dfrac{R}{4} + \dfrac{3R}{4} = R$. Sine $\widetilde{F}z, \widetilde{F}x_0 \in C(K)$, F for $O(x)$ be the nonexpansive type fuzzy mappings, by Nadler theorem, $\forall\ a \in \widetilde{F}z\ \exists\ b \in \widetilde{F}x_0$ such that $\| a - b \| \leqslant H(\widetilde{F}z, \widetilde{F}x_0) \leqslant \| z - x_0 \| \leqslant R$, therefore $a \in S$,we have $\widetilde{F}z \subseteq S$.

If $z \in S$ and $z \notin GL(x_0,\overline{Co}\widetilde{F}x_0,K)$,then $\exists\ v \in \overline{Co}\widetilde{F}x_0$ such that $\| z - v \| \leqslant R$ and $\| z - v \| \geqslant \| z - x_0 \|$,thus $H(\widetilde{F}z,\widetilde{F}x_0) \leqslant \| z - x_0 \| \leqslant \| z - v \| \leqslant R, \forall\ a \in \widetilde{F}z$ by Nadler theorem $\exists\ b \in \widetilde{F}x_0$ such that $\| a - b \| \leqslant H(\widetilde{F}z,\widetilde{F}x_0) \leqslant R$, therefore $a \in S$,we have $\widetilde{F}z \subseteq S$.

By S is nonempty,closed,convex and bounded, $\widetilde{F}: S \to 2^S$ is the multivalued nonexpansive point-compact self-mappings and X be a Banach space whose bounded closed convex subsets have the fixed point property for these pappings,we have $p \in S \subseteq K$ such that $p \in \widetilde{F}p$,i. e. , $Fp(p) \geqslant O(p)$.

In particular, if $O(x) = \max\limits_{u \in K} Fx(u)$, we have $Fp(p) \geqslant \max\limits_{u \in K} Fp(u) \geqslant Fp(p)$, therefore $Fp(p) = \max\limits_{u \in K} Fp(u)$, i. e. , p be a fixed point of F.

COROLLARY 2. 2. Let Z,K satisfy the conditions of theorem 2. 1, let $T: K \to C(K)$ nonexpansive. If there exists $x_0 \in K$ such that $GL(x_0,\overline{Co}Tx_0, K)$ is bounded then exsits $p \in K, p \in Tp$,i. e. , p is a fixed point of T .

PROOF. From $T: K \to C(K)$, we can define that $F: K \to \mathscr{F}(K)$ such that $\forall\ x \in K$,

$$Fx(u) = \begin{cases} 1, u \in Tx \\ 0, u \notin Tx \end{cases} \qquad \forall\ u \in K$$

and let $O(x) \equiv 1, \forall\ x \in K$, then F for $O(x)$ satisfies the conditions (2) in theorem 2. 1,moreover $Tx_0 = \widetilde{F}x_0$,by theorem 2. 1,conclution holds.

By the corollary 2. 2,we have.

COROLLARY 2. 3. Let X,K satisfy the conditions of theorem 2. 1, let $f: K \to K$ single-valued nonexpansive, if there exists $x_0 \in K$ such that $GL(x_0, fx_0, K)$ is bounded,then there exsists $p \in K, p$ is a fixed point of f .

REMARK 2. 4. If is obvious $GL(x,y,K) = \{z \in K \mid \| z - y \| < \| z - x \| \} \subseteq \{z \in K \mid \| z - y \| \leqslant \| y - x \| \} = G(x,y,K)$, and it is easy to see that $GL(x,y,K) \subseteq \{z \in K \mid \tau(x - z, y - z) < 0\} = LS(x,y,K)$, hence the corollary 2. 2 improves [4,Th. 1],[5. Th. 1] and [6. Th. 2],the corollary 2. 3 improves and generalizes the corresponding results of [1,2,3], theorem 2. 1 unifies and improves the corresponding results in [1, 2, 3, 4, 5, 6].

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