ON SOME TYPES OF FUZZY CONTINUOUS MULTIFUNCTIONS

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Abstract: The concept of fuzzy θ-continuos and fuzzy weakly δ-continuous multifunctions are introduced and studied in the light of q-coincidence. Later it has been presented some counter-examples related to the these fuzzy multifunctions.

Keywords: Fuzzy lower and upper θ-continuous; fuzzy lower and upper weakly δ-continuous; fuzzy lower and upper inverse of a fuzzy multifunction.

1. Introduction

The concept of fuzzy multifunctions was introduced and extended to the concepts of fuzzy upper and lower semi-continuous multifunctions by Papageorgiou [6]. In [5] Mukherjee and Malakar introduced and investigated fuzzy almost continuity and fuzzy weakly continuity on fuzzy multifunctions by using the concept of quasi-coincidence of Pu and Liu[7,8]. In this paper fuzzy θ-contunity and fuzzy weakly δ-contunity given by Mukherjee and Sinha[4] are extended to fuzzy multifunctions. Also it is shown that fuzzy θ-continuous multifunctions given by the author, fuzzy semi-continuous multifunctions given by Mukherjee and Sinha [4] are independent of each other.

2. Preliminaries

The definitions of fuzzy sets, fuzzy point, fuzzy topology, fuzzy regular open (closed) sets, concept of quasi-coincidence and other related concepts can be found in [1,2,7,8,9]. For the concept of fuzzy δ-closed(open) set, fuzzy neigbourhood (nbd for short), fuzzy θ-nbd and some theorems see [4] and the definitions of fuzzy extremally disconnected spaces can be found in [3].

Throughout the paper, by (X,τ) or simply by X we will mean a topological space in the classical sense , and (Y,τ_Y) or simply Y will stand for a fuzzy topological space (fts for short) as defined by Chang [2]. Fuzzy sets of Y will be denoted by the α,β,μ ...etc. $\mathrm{Int}\mu$, $\mathrm{Cl}\mu$ and $\mu\,' = 1-\mu$ will denote respectively the interior, closure and complement of fuzzy set.

Definition 2.1 [6]. Let (X,τ) be a topological space in the classical sense and (Y,τ_Y) be an fts. $F:X\to Y$ is called a fuzzy multifunctions iff for each $x\in X$, $F(x)$ is a fuzzy set in Y.

Throughout the paper, unless otherwise stated by $F:X\to Y$ we will mean that F is a fuzzy multifunction from a classical topological space (X,τ) to an fts (Y,τ_Y).

Definition 2.2 [5]. For a fuzzy multifunction $F:X\to Y$, the upper inverse $F^+(\mu)$ and lower inverse $F^-(\mu)$ of a fuzzy set μ in Y are defined as follows: $F^+(\mu) = \{ x\in X : F(x)\leq\mu \}$, $F^-(\mu) =\{ x\in X :F(x)q\mu \}$.

Theorem 2.3 [5]. For a fuzzy multifunction $F:X\to Y$, we have $F^-(1-\mu)= X / F^+(\mu)$, for any fuzzy set μ in Y.

Definition 2.4 [5]. A fuzzy multifunction $F:X\rightarrow Y$ is called

a) fuzzy lower semi-continuous (f.l.s.c., in short) at a point $x_0 \in X$ iff for every fuzzy open set μ in Y with $x_0 \in F^-(\mu)$, there exists an open nbd U of x_0 in X such that $U \subset F^-(\mu)$,i.e., $F(x)q\mu$ for each $x \in U$;

b) fuzzy upper semi-continuous (f.u.s.c., in short) at a point $x_0 \in X$ iff for every fuzzy open set μ in Y with $x_0 \in F^+(\mu)$, there exists an open nbd U of x_0 in X such that $U \subset F^+(\mu)$,i.e., $F(x)q\mu$ for each $x \in U$;

c) f.l.s.c. (f.u.s.c.) on X iff it is respectively so at each $x_0 \in X$.

Theorem 2.5 [5]. A fuzzy multifunction $F:X\rightarrow Y$ is f.l.s.c. iff for any fuzzy open set β in Y, $F^-(\beta)$ is open in X.

Theorem 2.6 [6]. A fuzzy multifunction $F:X\rightarrow Y$ is f.u.s.c. iff for any fuzzy open set β in Y, $F^+(\beta)$ is open in X.

Definition 2.7 [5]. A fuzzy multifunction $F:X\rightarrow Y$ is called

a) fuzzy lower almost continuous (f.l.a.c.,in short) at some point $x_0 \in X$ iff for every fuzzy open set μ in Y with $x_0 \in F^-(\mu)$,there exists an open nbd U of x_0 such that $U \subset F^-(\mathrm{IntCl}\mu)$;

b) fuzzy upper almost continuous (f.u.a.c., in short) at some point $x_0 \in X$ iff for every fuzzy open set μ in Y with $x_0 \in F^+(\mu)$, there exists an open nbd U of x_0 such that $U \subset F^+(\mathrm{IntCl}\mu)$;

c) f.l.a.c. (f.u.a.c.) on X iff F is respectively so at each $x_0 \in X$.

Definition 2.8[5]. A fuzzy multifunction $F:X\rightarrow Y$ is said to be

a) fuzzy lower weakly continuous (f.l.w.c., in short) at some point $x_0 \in X$ iff for every fuzzy open set μ in Y with $x_0 \in F^-(\mu)$,there exist an open nbd U of x_0 such that $U \subset F^-(\mathrm{Cl}\mu)$;

b) fuzzy upper weakly continuous (f.u.w.c.,in short) at some point $x_0 \in X$ iff for every fuzzy open set μ in Y with $x_0 \in F^+(\mu)$, there exist an open nbd U of x_0 such that $U \subset F^+(\mathrm{Cl}\mu)$;

c) f.l.w.c. (f.u.w.c.) on X iff F is respectively so at each $x_0 \in X$.

Definition 2.9. A fuzzy multifunction $F:X\rightarrow Y$ is called

a) fuzzy lower θ-continuous (f.l.θ-c., in short) at a point $x_0 \in X$ iff for every fuzzy upper set μ in Y with $x_0 \in F^-(\mu)$,there exists an open nbd U of x_0 such that $\mathrm{Cl}U \subset F^-(\mathrm{Cl}\mu)$;

b) fuzzy upper θ-continuous (f.u.θ-c., in short) at a point $x_0 \in X$ iff for every fuzzy open set μ in Y with $x_0 \in F^+(\mu)$, there exists an open nbd U of x_0 such that $\mathrm{Cl}U \subset F^+(\mathrm{Cl}\mu)$;

c) f.l.θ-c. (f.u.θ-c) on X iff F is respectively so at each $x_0 \in X$.

Theorem 2.10. If $F:X\rightarrow Y$ is an f.l.θ-c. multifunction, then the followings are true:

a) $[F^+(\beta)]_\theta \subset F^+([\beta]_\theta)$, for every fuzzy set β in Y.

b) $[F^+(\mu)]_\theta \subset F^+(Cl\mu)$,for every fuzzy open set μ in Y.

c) For each fuzzy θ-closed set γ in Y, $F^+(\gamma)$ is θ-closed in X.

d) For each fuzzy θ-open set λ in Y, $F^-(\lambda)$ is θ-open in X.

Proof.(a): Let $x \in [F^+(\beta)]_\theta$ and let μ be any open q-nbd of y_α for which $y_\alpha \in F(x)$. By f.l.θ-continuity of F, there exist an open nbd U of x such that $F(z)qCl\mu$ for all $z \in ClU$. Since $x \in [F^+(\beta)]_\theta$, there exist $z_o \in X$ such that $z_o \in ClU \cap F^+(\beta)$. If $z_o \in ClU$ then $F(z_o)qCl\mu$ and if $z_o \in F^+(\beta)$ then $F(z_o)qCl\mu$ and if $z_o \in F^+(\beta)$ then $F(z_o) \leq \beta$. Thus $\beta qCl\mu$ and $y_\alpha \in [\beta]_\theta$. Then $x \in F^+([\beta]_\theta)$ and hence $[F^+(\beta)]_\theta \subset F^+([\beta]_\theta)$.

(b): Since μ is fuzzy open in Y, $Cl\mu=[\mu]_\theta$ and we have from (a) $F^+(\beta)]_\theta \subset F^+([\mu]_\theta)= F^+(Cl\mu)$.

(c): Let γ be a fuzzy closed in Y. We have $\gamma = [\gamma]_\theta$. By (a),

$[F^+(\gamma)]_\theta \subset F^+([\gamma]_\theta) = F^+(\gamma) \Rightarrow [F^+(\gamma)]_\theta = F^+(\gamma)$ and the result follows.

(d): Straightforward.

Theorem 2.11. If $F:X \to Y$ is f.l.θ-c., then for each fuzzy set μ in Y, $F^- [Int(\gamma_\theta)] \subset Int[F^-(\gamma)]_\theta$.

Proof. Obvious.

Definition 2.12. A fuzzy multifunction $F:X \to Y$ is called

a)fuzzy lower weakly δ-continuous (f.l.w.δ-c.,in short) at some point $x_0 \in X$ iff for every fuzzy open set μ in Y with $x_0 \in F^-(\mu)$, there exists an open nbd U of x_0 such that $IntClU \subset F^-(Cl\mu)$;

b)fuzzy upper weakly δ-continuous (f.u.w.δ-c., in short) at some point $x_0 \in X$ iff for every fuzzy open set μ in Y with $x_0 \in F^+(\mu)$, there exists an open nbd U of x_0 such that $IntClU \subset F^+(Cl\mu)$;

c) f.l.w.δ-c.(f.u.w.δ-c) on X iff F is respectively so at each $x_0 \in X$.

Theorem 2.13. For a fuzzy multifunction $F:X \to Y$ the following statements are equivalent

a) F is f.l.w.δ-c.

b) $[F^+(\beta)]_\delta \subset F^+([\beta]_\theta)$, for every fuzzy set β in Y.

c) $F^+(\beta)$ is δ-closed in X, for every fuzzy θ-closed set β in Y.

d) $F^-(\beta)$ is δ-open in X, for every fuzzy θ-open set β in Y.

e) $F^-(\lambda)$ is δ-open in X, for every fuzzy regular open set λ in Y.

f) For each point x of X and each fuzzy θ-nbd μ of y_α fuzzy point for which $y_\alpha \in F(x)$, then $F^-(\mu)$ is a δ-nbd of x.

Proof. (a)$\Rightarrow$(b): Let $x \in [F^+(\beta)]_\delta$ and $y_\alpha \in F(x)$. μ is fuzzy open set of y_α fuzzy point such that $x \in F^-(\mu)$. Since F is f.l.w.δ-c., there exist an open nbd U of x such that $IntClU \subset F^-(ClU)$,i.e., $F(z)qCl\mu$ for all $z \in IntClU$. Since $x \in [F^+(\beta)]_\delta \Rightarrow IntClU \cap F^+(\beta) \neq \emptyset$. We have $z \in IntClU \cap F^+(\beta)$. If $z \in IntClU$ then $F(z)qCl\mu$ and if $z \in F^+(\beta)$ then $F(z) \leq \beta$. Thus $\beta qCl\mu$ and hence $y_\alpha \in [\beta]_\theta$. Then $x \in F^+([\beta]_\theta)$ and so $[F^+(\beta)]_\delta \subset F^+([\beta]_\theta)$.

(b) $\Rightarrow$ (c): Let β be a fuzzy θ-closed in Y,i.e., $\beta=[\beta]_\theta$. By hypothesis, $[F^+(\beta)]_\delta \subset F^+([\beta]_\theta) = F^+(\beta)$.Hence $F^+(\beta)$is δ-closed in X.

(c) $\Rightarrow$ (d): Let β be a fuzzy θ-closed in Y. Then 1-β is a fuzzy θ-closed in Y, hence

$F^+(1-\beta) = X - F^-(\beta)$ is δ-closed in X which implies $F^-(\beta)$ is δ-open in X.

(d) ⇒ (e): Let λ be fuzzy regular open in Y. Since every regular open set is fuzzy δ-open, by(d) $F^-(\lambda)$is δ-open in X.

(e) ⇒ (a): Let $x\in X$ and $y_\alpha \in F(x)$. μ any fuzzy open q-nbd of y_α fuzzy point. Also , $\mathrm{IntCl}\mu$ is fuzzy regular open q-nbd of y_α . Since $F^-(\mathrm{IntCl}\mu)$ is δ-open in X, there exist an open nbd U of x such that $x\in U\subset \mathrm{IntCl}U\subset F^-(\mathrm{IntCl}\mu)$.Hence $\mathrm{IntCl}U\subset F^-(\mathrm{Cl}\mu)$.

(a) ⇒ (f): Let $x\in X$ and $y_\alpha \in F(x)$.β is fuzzy θ-nbd of y_α fuzzy point. Then, There exist an open q-nbd μ of y_α such that $\mathrm{Cl}\mu \not{q} \beta'$.Since F is f.l.w.δ-c.,there exist an open nbd U of x such that $\mathrm{IntCl}U\subset F^-(\mathrm{Cl}\mu)\subset F^-(\beta) \Rightarrow \mathrm{IntCl}U\cap[F^-(\beta)]^c=\varnothing$. Hence $F^-(\beta)$ is a fuzzy δ-nbd of x.

(f) ⇒ (a): One can use a similar technique as in (a)⇒(f) .

Theorem 2.14. For a fuzzy multifunction $F:X\to Y$ the following statements are equivalent:

a) F is f.l.w.δ-c.

b) $[F^-(\mathrm{Int}[\beta]_\theta) \subset \mathrm{Int}[F^-(\beta)]_\delta$ for every β fuzzy set in Y.

c) $[F^-(\mu)]_\delta \subset F^+(\mathrm{Cl}\mu)$,for every μ fuzzy open set in Y.

d) $F^-(\mu) \subset \mathrm{Int}[F^-(\mathrm{Cl}\mu)]_\delta$, for every μ fuzzy open set in Y.

Proof.(a) ⇒ (b): Let be β fuzzy set in Y. For a 1-β fuzzy set in Y, by Theorem 2.13(b) we have

$$[F^+(1-\beta)]_\delta\subset F^+([1-\beta]_\theta) \Rightarrow \mathrm{Int}[F^-(\beta)]_\delta \subset F^-(\mathrm{Int}[\beta]_\theta).$$

(b) ⇒ (a) : Obvious.

(a) ⇒ (c): For a fuzzy open set μ in Y we have $\mathrm{Cl}\mu=[\mu]_\theta$. From this equality we obtain the required implication easily.

(c) ⇒ (d): Let μ be fuzzy open set in Y. Since 1-Clμ is a fuzzy open set in Y, we have $[F^+(1-\mathrm{Cl}\mu)]_\delta \subset F^+(\mathrm{Cl}(1-\mathrm{Cl}\mu)) \Rightarrow F^-(\mu)\subset \mathrm{Int}[F^-(\mathrm{Cl}\mu)]_\delta$.

(d)⇒ (a): For an arbitrary $x\in X$ and arbitrary fuzzy open set μ in Y with $x\in F^-(\mu)$.By(d),we have $F^-(\mu)\subset \mathrm{Int}[F^-(\mathrm{Cl}\mu)]_\delta$. Hence, there exist an open nbd U of x such that $\mathrm{IntCl}U\subset F^-(\mathrm{Cl}\mu)$, then F is f.l.w.δ-c.

3. Mutual Relationship

In section 2 we have observed the following implications diagram:

$$\begin{tikzcd}
\text{f.l(u).s.c.} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow, "/" marking] & \text{f.l(u).a.c.} \arrow[r, Rightarrow] & \text{f.l(u).w.c.} \\
\text{f.l(u).}\theta\text{-c.} \arrow[rr, Rightarrow] & & \text{f.l(u).w.}\delta\text{-c.} \arrow[u, Rightarrow]
\end{tikzcd}$$

We now show by means of the following examples that none of the above implications can be reserved, in general. In these examples we use the notation $C_\alpha(0\le\alpha\le1)$ to denote the constant fuzzy set such that $C_\alpha(y)=\alpha$, for all $y\in Y$.

Example 3.1. Let $X=\{a,b\}$,$Y=[\,0,1]$, $\tau=\{X,\varnothing,\{a\}\}$,$\tau_Y=\{C_0,C_1,C_{2/5}\}$ and let $F:(X,\tau)\to(Y,\tau_Y)$ be given by $F(a)=C_{1/2}$,$F(b)=C_{11/12}$. It is obvious that F is f.l.θ-c.,but F is not f.l.s.c. because by Theorem 2.5.,for

$C_{1/5} \in \tau_Y$, $F^-(C_{2/5})=\{b\} \notin \tau$.

Example 3.2. Let X and Y be the same as in Example 3.1 and be $\tau=\{X,\emptyset,\{b\}\}, \tau_Y=\{C_0,C_1,C_{1/8},C_{3/4}\}$.Consider the fuzzy multifunctions $F:(X,\tau)\to(Y,\tau_Y)$ as follows $F(a)=C_{1/8}$, $F(b)=C_{1/2}$. F is f.u. θ–c,but since $F^+(C_{1/8})=\{a\} \notin \tau$ for $C_{1/8} \in \tau_Y$, F is not f.u.s-c.

Example 3.3. Let (X,τ) and Y be the same as in Example 3.2.and be $\tau_Y=\{C_0,C_1,C_{1/2}\}$. We define the fuzzy multifunction $F:(x,\tau)\to(Y,\tau_Y)$ by letting $F(a)=C_{1/2}$,$F(b)=C_{5/6}$. Then F is f.l.s.c but not f.l.θ-c.

Example 3.4. Let X={a,b} and Y=[0,1]. Let τ and τ_Y be respectively the topology on X and fuzzy topology on Y given by $\tau = \{X,\emptyset, \{a\}\}$ and $\tau_Y = \{C_0,C_1,C_{1/3}\}$. It is obvious that F is f.u.θ-.c., but it is not f.u.s.c.

Examples 3.1., 3.2., 3.3. and 3.4. establish the following:

Theorem 3.5. Fuzzy lower (upper) semi continuity and fuzzy lower (upper) θ-continuity are independent notions.

Example 3.6. Let (X,τ) be as described in Example 3.5 and take $\tau_Y=\{C_0,C_1,C_{1/3},C_{2/3}\}$ on Y=[0,1].Fuzzy multifunction $F:(X,\tau)\to(Y,\tau_Y)$ defined by $F(a)=C_{5/6}$, $F(b)=C_{1/2}$,$F(c)=C_{1/4}$.

From this example we can give:

A fuzzy multifunction F f.l.w.δ-c. mapping need not be f.l.θ-c. mapping.

Example 3.7. Let X ={a,b,c} and Y=[0,1]. Let τ and τ_Y be respectively the topology on X and fuzzy topology on Y given by $\tau=\{X,\emptyset,\{b\},\{c\},\{b,c\}\}$,$\tau_Y = \{C_0,C_1,C_{1/5},C_{5/8}\}$.

Define the fuzzy multifunction $F:(X,\tau)\to(Y,\tau_Y)$ as follows $F(a)=C_{1/2}$,$F(b)=C_{1/6}$, $F(c)=C_{1/5}$.

Clearly F is f.u.w.δ–c but not f.u. θ–c.

Theorem 3.8. If a fuzzy multifunction $F:X\to Y$ is fuzzy lower (upper) weakly δ-continuous and X extremally disconnected, then F is fuzzy lower (upper) θ-continuous.

Proof. Obvious.

Example 3.9. Let(X,τ) be the same as in Example 3.7. and let be let $\tau_Y = \{C_0,C_1,C_{2/5},C_{1/2}\}$ on Y=[0,1].

Define the fuzzy multifunction $F:(X,\tau)\to(Y,\tau_Y)$ as follows $F(a)=C_{7/8}$,$F(b)=C_{1/3}$. It is clear that F is f.l.w.c. but not f.l.w.δ-c.

Example 3.10. Let X={a,b,c},$\tau=\{X,\emptyset,\{a\},\{b\}\}$ and let (Y,τ_Y) be the same as in Example 3.7. Consider the fuzzy multifunction $F:(X,\tau)\to(Y,\tau_Y)$ by letting $F(a)=C_{1/2}$,$F(b)=C_{2/3}$, $F(c)=C_{11/12}$.

It is easy to see that F is f.u.w.c.but not f.u.w.δ-c.

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