Some Results of Hom Functor in Categories of F_R^A-modules

Zhao Jianli

Department of Mathematics, Liaocheng Teachers College,

Shandong, 252059, P. R. China

Abstract In this paper, we are to discuss the properties of Hom functor in categories of F_R^A-modules and give the some difference between the Hom functor in categories of F_R^A-modules and the Hom functor in categories of R-modules

Keywords F_R^A-module, F-homomorphism, category of F_R^A-modules, category of F-abel groups, F-exact sequence.

1. Introduction

[1-3] established the basic knowlege of theory of F_R^A-module, and provied the idea researching fuzzy ring from outside. In this paper, we will carry on the work of [1-3], study the properties of Hom functor in categories of F_R^A-modules.

Let X be a nonempty set, L be a complete distributive lattice (with 0 and 1), a fuzzy subset A on X is characterised by a mapping $A: X \to L$. X^L denotes the set of whole fuzzy subset of X. In this paper, R is a ring with identity $1 \neq 0$ and module which iovolved is an unitary left R-module.

Definition 1.1. Let R is a ring, $A \in R^L$, if for all $x, y \in R$, we have:

1) $A(x-y) \geqslant A(x) \wedge A(y)$;

2) $A(xy) \geqslant A(x) \wedge A(y)$;

3) $A(0) = 1$,

then A is called a fuzzy subring of R .

Definition 1.2. [1] Let M be a left R-module, A a fuzzy subirng of R, $B_M \in M^L$, if for all $x, y \in M, r \in R$, we have

1) $B_M(x-y) \geqslant B_M(x) \wedge B_M(y)$,

2)$B_M(0)=1$,

3)$B_M(rx)\geqslant A(r)\wedge B_M(x)$,

then B_M is called an F_R^A-submodule (or F_R^A-module).

Definition 1.3. Let N be a left R-submodule of M, B_M and C_N be F_R^A-submodule of M and N respectively, if for all $x\in N$, we have

$$B_M(x)\geqslant C_N(x)$$

then C_N is called an F-submodule of B_M.

Definition 1.4. Let M and N be two R-modules, $f: M\to N$ be an R-homomorphism, B_M be an F_R^A-submodule of M, F_R^A-submodule $\tilde{f}(B_M)$ of N is defined by

$$\tilde{f}(B_M)(y)=\begin{cases} \vee\{B_M(x)\mid x\in M, f(x)=y\}, \text{if } f^{-1}\neq\Phi, \\ 0, \quad \text{is } f^{-1}(y)=\Phi, \end{cases}$$

for all $y\in N$.

Definition 1.5. Let M and N be two left R-module, $f: M\to N$ be an R-homomorphism, B_M and C_N be F_R^A-submodule of M and N, respectively, if $\tilde{f}(B_M)\leqslant C_N$, then $\tilde{f}$ is an F-homomorphism from B_M into C_N, writes by $\tilde{f}: B_M\to C_N$,

Definition 1.6 The category of F_R^A-modules F_R^A-Mod is defined by:

1)Objects are all F_R^A-modules,

2)For all B_M, $C_N\in \mathrm{Obj}$ (F_R^A-Mod), the set of morphisms is

Hom $(B_M, C_N)=\{\tilde{f}\mid \tilde{f}$ is an arbityary F-homomorphism from B_M into $C_N\}$,

3) For all $\tilde{f}\in \mathrm{Hom}(B_M, C_N)$, $\tilde{g}\in \mathrm{Hom}(C_N, D_S)$, the composition of $\tilde{f}$ and $\tilde{g}$ is defined by $\tilde{f}\tilde{g}=\widetilde{fg}$.

Definition 1.7. Let G is an abel group, $B_G\in G^L$, if for all $x, y\in G$, we have

1)$B_G(x-y)\geqslant B_G(x)\wedge B_G(y)$,

2)$B_G(0)=1$,

then B_G is called an F-subgroup of G or F-abel group.

Definition 1.8. The category of F-abel groups F-AG is defined by:

1)Obiects are all F-abel groups,

2)For all B_G, $C_H\in \mathrm{Obj}(\text{F-AG})$, the morphims are

$$\mathrm{Hom}\ (B_G, C_H)=\{\tilde{f}\mid \tilde{f}: B_G\to C_H \text{ is an F-homomorphism}\}$$

3)For all $\tilde{f}\in \mathrm{Hom}(B_G, C_H)$, $\tilde{g}\in \mathrm{Hom}(C_H, D_N)$, the composition of $\tilde{f}$ and $\tilde{g}$ is defined by $\tilde{f}\tilde{g}=\widetilde{fg}$.

Defintion 1.9. Let $\{M_i\}_{i\in I}$ is a collection of R-modules, $B^i_{M_i}$ is an F_R^A-module for all $i\in I$, the fuzzy subset $\bigoplus_{i\in I} B^i_{M_i}$ of $\bigoplus M_i$ and fuzzy subset $\prod_{i\in I} B^i_{M_i}$ of $\prod_{i\in I} M_i$ are defined by

$(\bigoplus_{i\in I} B^i_{M_i})x = \wedge \{B^i_{M_i}(x_i) \mid i \in I\}$, for all $x = \langle x_i \rangle \in \bigoplus_{i\in I} M_i$,

$(\prod_{i\in I} B^i_{M_i})x = \wedge \{B^i_{M_i}(x_i) \mid i \in I\}$, for all $x = \langle x_i \rangle \in \prod_{i\in I} M_i$, respectively. Then $\bigoplus_{i\in I} B^i_{M_i}$ and $\prod_{i\in I} B^i_{M_i}$ are called the fuzzy external direct sum and the fuzzy direct product of $\{B^i_{M_i}\}_{i\in I}$, respectively.

Definition 1. 10. A sequence of $F^{\wedge}_R$-modules and F-homomorphisms:

$$\cdots \to B^{i-1}_{M_i} \xrightarrow{\tilde{f}_{i-1}} B^i_{M_i} \xrightarrow{\tilde{f}_i} B^{i+1}_{M_i} \to \cdots$$

is said to be F-exact sequence if

$$\operatorname{im} \tilde{f}_{i-1} = \ker \tilde{f}_i$$

for every i.

Proposition 1. 11. [2] Let $\{B^i_{M_i}\}_{i\in I}$ is a collection of category $F^{\wedge}_R$-Mod, then $\prod_{i\in I} B^i_{M_i}$ and $\bigoplus_{i\in I} B^i_{M_i}$ are product and coproduct of $\{B^i_{M_i}\}_{i\in I}$ in category $F^{\wedge}_R$-Mod, respectively.

2. Some results of Hom functor in $F^{\wedge}_R$-Mod

Let B_M and C_N be $F^{\wedge}_R$-submodule of M and N, respectively, for all $\tilde{f}, \tilde{g} \in \operatorname{Hom}(B_M, C_N)$, let

$$\tilde{f} + \tilde{g} = \widetilde{f+g} \qquad (1)$$

then (1) is an algebra operation of Hom (B_M, C_N), and it is easy to prove the following Proposition 2. 1.

Proposition 2. 1. (Hom (B_M, C_N), +) is an abel group.

Theorem 2. 2. Let $\{B^i_{M_i}\}_{i\in I}$ is a collection of $F^{\wedge}_R$-modules, then there exist the following isomorphism of group,

1) $\operatorname{Hom}(\bigoplus_{i\in I} B^i_{M_i}, B_M) \cong \prod_{i\in I} \operatorname{Hom}(B^i_{M_i}, B_M)$,

2) $\operatorname{Hom}(B_M, \prod_{i\in I} B^i_{M_i}) \cong \prod_{i\in I} \operatorname{Hom}(B_M, B^i_{M_i})$

for all $F^{\wedge}_R$-module B_M.

Proof. 1) Let $\tilde{\theta}$: $\operatorname{Hom}(\bigoplus_{i\in I} B^i_{M_i}, B_M) \to \operatorname{Hom}(B^i_{M_i}, B_M)$ such that

$$\theta(\tilde{f}) = \{\tilde{f}\tilde{\pi}_i\}_{i\in I}$$

for all $\tilde{f} \in \operatorname{Hom}(\bigoplus_{i\in I} B^i_{M_i}, B_M)$, where π_i is ith canonical injection of M_i into $\bigoplus_{i\in I} M_i$ (for all $i \in I$). It is clear that θ is a homomorphism of abel group. To show that θ is surjective, let $\{\tilde{g}_i\}_{i\in I} \in$

$\prod_{i\in I}$ Hom $(B^{i}_{M_i}, B_M)$, there is an F-homomorphism

$$\tilde{\eta}: \bigoplus_{i\in I} B^{i}_{M_i} \to B_M$$

such that the diagram 1

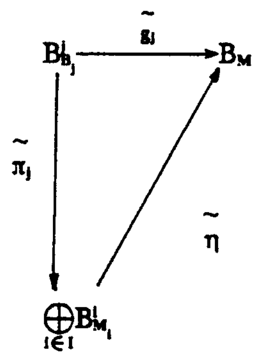

diagram 1

is commutative (by Proposition 1. 11), for all $j\in I$. Since then

$$\theta(\tilde{f}) = \{\tilde{\eta}\tilde{\pi}_i\}_{i\in I} = \{\tilde{g}_i\}_{i\in I},$$

we see that $\tilde{\theta}$ is surjective. To show θ is also injective, let $\tilde{\alpha}\in \mathrm{Ker}\theta$, then

$$\theta(\tilde{\alpha}) = \tilde{0} = \{\tilde{\alpha}\tilde{\pi}_i\}_{i\in I},$$

and the diagram 2

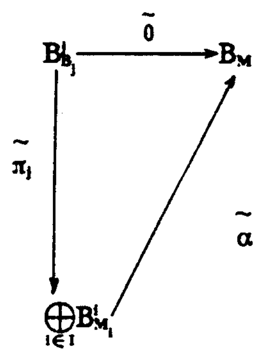

diagram 2

is commutative (in which 0 denotes the zero R-homomorphism), for all $j\in I$.

Now since $(\bigoplus_{i\in I} B^{i}_{M_i}, \{\tilde{\pi}\}_{i\in I})$ is a coproduct of $\{B^{i}_{M_i}\}_{i\in I}$ and since the zero F-homomorphism $\tilde{0}$ from $\bigoplus_{i\in I} B^{i}_{M_i}$ into B_M also makes the diagram 2 commutative, by Proposition 1. 11, we have $\tilde{\alpha} = \tilde{0}$, whence θ is also injective.

2)This is the dual of 1).

Corollary 2. 3. If R is commutative then the above isomorphisms are R-module isomorphisms.

In fact, for all $\tilde{f}\in$ Hom (B_M, C_N), $r\in R$, Let:

$$r\,\tilde{f} = r\tilde{f}$$

It is clear that $Hom(B_M, C_N)$ is a left R-module, hence we can prove that the isomorphisms of abel group in Theorem 2. 2 is left R-module isomorphisms.

Corollary 2. 4. If I is finite, then

1) $Hom(\bigoplus_{i\in I} B^i_{M_i}, B_M) \cong \bigoplus_{i\in I} Hom(B^i_{M_i}, B_M)$,

2) $Hom(B_M, \bigoplus_{i\in I} B^i_{M_i}) \cong \bigoplus_{i\in I} Hom(B_M, B^i_{M_i})$.

Let $B^1_{M_1}, B^2_{M_2}$ are two F^A_R-module, $\tilde{f} \in Hom(B^1_{M_1}, B^2_{M_2})$, if B_M is an arbitrary F^A_R-module, then we can define homomorphism of abel group

$$\tilde{f}_* : Hom(B_M, B^1_{M_1}) \to Hom(B_M, B^2_{M_2}),$$

by the assignment

$$\tilde{f}_* : \tilde{\theta} \to \tilde{f}_*(\theta) = \tilde{f}\tilde{\theta} = \tilde{f\theta},$$

we say that $\tilde{f}_*$ is induced by f.

Simillarly, we can define homomorphism of abel group

$$\tilde{f}_* : Hom(B^2_{M_2}, B_M) \to Hom(B^1_{M_1}, B_M)$$

by the assignment

$$\tilde{f}^* : \tilde{\theta} \to \tilde{f}^*(\tilde{\theta}) = \tilde{\theta}\tilde{f} = \tilde{\theta f},$$

we also say that $\tilde{f}^*$ is induced by $\tilde{f}$.

Theorem 2. 5. Let $B^1_{M_1}, B^2_{M_2}, B^3_{M_3}$ are F^A_R-module, $\tilde{f}, \tilde{h} \in Hom(B^1_{M_1}, B^2_{M_2})$, $\tilde{g} \in Hom(B^2_{M_2}, B^3_{M_3})$, then we have

(1) $(\tilde{g}\tilde{f})_* = \tilde{g}_*\tilde{f}_*$, (2) $(\tilde{g}\tilde{f})^* = \tilde{f}^*\tilde{g}^*$,

(3) $(\tilde{f} + \tilde{h})_* = \tilde{f}^* + \tilde{h}^*$, (4) $(\tilde{f} + \tilde{h})^* = \tilde{f}^* + \tilde{h}^*$.

Proof. The proof is easy, and hence omitted.

In the theory of R-module, it is well-known that:

1) If $0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ is any exact sequence of R-modules, M is an R-module, then the following sequence

$$0 \to Hom(M, M_1) \xrightarrow{f_*} Hom(M, M_2) \xrightarrow{g_*} Hom(M_1, M_3)$$

is exact.

2) If $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$ is any exact sequence of R-modules, M is an R-module, then the following sequence

$$0 \to \mathrm{Hom}(M_3, M) \xrightarrow{g^*} \mathrm{Hom}(M_2, M) \xrightarrow{f^*} \mathrm{Hom}(M_1, M)$$

is exact.

But the following examples show that these do not hold in the case of fuzziness.

Example Let M be a nonozero R-module, $L=[0,1]$, and the fuzzy subring A of R and F_R^A-module B_M, B_M^1, B_M^2, B_M^3, are defined by:

$$A(r)=\begin{cases} 1/8 & \text{if } r\in R \text{ and } r\neq 0, \\ 1 & \text{if } r\in R \text{ and } r=0; \end{cases}$$

$$B_M(x)=\begin{cases} 1/5 & \text{if } x\in M \text{ and } x\neq 0, \\ 1 & \text{if } x\in M \text{ and } x=0; \end{cases}$$

$$B_M^1(x)=\begin{cases} 1/6 & \text{if } x\in M \text{ and } x\neq 0, \\ 1 & \text{if } x\in M \text{ and } x=0; \end{cases}$$

$$B_M^2(x)=\begin{cases} 1/3 & \text{if } x\in M \text{ and } x\neq 0, \\ 1 & \text{if } x\in M \text{ and } x=0; \end{cases}$$

$$B_M^3(x)=\begin{cases} 1/2 & \text{if } x\in M \text{ and } x\neq 0, \\ 1 & \text{if } x\in M \text{ and } x=0; \end{cases}$$

it is easy to prove that the fuzzy sequence

$$\bar{0} \to B_M^1 \xrightarrow{\tilde{f}} B_M^2 \xrightarrow{\tilde{g}} B_M^3$$

is an F-exact sequence of F_R^A-modules, where f is the identity map and g is the zero map on M. But the sequence $0 \to \mathrm{Hom}(B_M, B_M^1) \xrightarrow{\tilde{f}_\cdot} \mathrm{Hom}(B_M, B_M^2) \xrightarrow{\tilde{g}_\cdot} \mathrm{Hom}(B_M, B_{M_3}^3)$ is not exact, because $\tilde{1} \in \ker \tilde{g}_\cdot$, and there is no $\tilde{\varphi} \in \mathrm{Hom}(B_M, B_M^2)$ such that $\tilde{f}_\cdot \tilde{\varphi} = \tilde{1}$, hence $\ker \tilde{g}_\cdot \neq \mathrm{im}\, \tilde{f}_\cdot$.

2) Let M be a nonzero R-module, the subring A of R and F_R^A-module B_M, B_M^1, B_M^2, B_M^3 are defined by

$$A(r)=\begin{cases} 1/8 & \text{if } r\in R \text{ and } r\neq 0, \\ 1 & \text{if } r\in R \text{ and } r=0; \end{cases}$$

$$B_M(x)=\begin{cases} 1/3 & \text{if } x\in M \text{ and } x\neq 0, \\ 1 & \text{if } x\in M \text{ and } x=0; \end{cases}$$

$$B_M^1(x)=\begin{cases} 1/4 & \text{if } x\in M \text{ and } x\neq 0, \\ 1 & \text{if } x\in M \text{ and } x=0; \end{cases}$$

$$B_M^2(x)=\begin{cases}1/4 & \text{if } x\in M \text{ and } x\neq 0,\\ 1 & \text{if } x\in M \text{ and } x=0;\end{cases}$$

$$B_M^3(x)=\begin{cases}1/2 & \text{if } x\in M \text{ and } x\neq 0,\\ 1 & \text{if } x\in M \text{ and } x=0;\end{cases}$$

it is easy to prove the following fuzzy sequence of F_R^A-modules

$$B_M^1 \xrightarrow{\tilde f} B_M^2 \xrightarrow{\tilde g} B_M^3 \to \bar 0$$

is exact, where f is the zero map, g is the identity map. But the sequence $0\to \mathrm{Hom}\,(B_M^3, B_M) \xrightarrow{\tilde g^*} \mathrm{Hom}\,(B_M^2, B_M) \xrightarrow{\tilde f^*} \mathrm{Hom}(B_M^1, B_M)$ is not exact, because $\tilde 1 \in \ker \tilde f^*$, but $\tilde 1 \overline{\in} \operatorname{im} \tilde g^*$ so $\ker \tilde f^* \neq \operatorname{im} \tilde g^*$.

Theoem 2. 6. Let B_M be an arbitrary F_R^A-module, then

1) If $\bar 0 \to B_{M_1}^1 \xrightarrow{\tilde f} B_{M_2}^2 \xrightarrow{\tilde g} B_{M_3}^3$ is an F-exact sequence of F_R^A-modules, then $0\to \mathrm{Hom}\,(B_M, B_M^1) \xrightarrow{\tilde f_*} \mathrm{Hom}\,(B_M, B_M^2) \xrightarrow{\tilde g_*} \mathrm{Hom}(B_M, B_{M_3}^3)$ is exact iff for any $\alpha\in \mathrm{Hom}(M, M_1)$, if $\widetilde{f\alpha}\in \operatorname{Ker} \tilde g_*$, we have $\tilde\alpha \in \mathrm{Hom}(B_M, B_{M_1}^1)$,

2) If $B_{M_1}^1 \xrightarrow{\tilde f} B_{M_2}^2 \xrightarrow{\tilde g} B_{M_3}^3 \to \bar 0$ is an F-exact sequence of F_R^A-modules, then $\mathrm{Hom}\,(B_{M_3}^3, B_M) \xrightarrow{\tilde g_*} \mathrm{Hom}\,(B_{M_2}^2, B_M) \xrightarrow{\tilde f_*} \mathrm{Hom}\,(B_{M_1}^1, B_M) \longrightarrow \bar 0$ is exact iff for any $\beta\in \mathrm{Hom}(M_3, M)$, if $\widetilde{g\beta}\in \ker \tilde f^*$, we have $\tilde\beta \in \mathrm{Hom}\,(B_{M_3}^3, B_M)$.

Proof 1) Necessity. If $0\to \mathrm{Hom}\,(B_M, B_{M_1}^1) \xrightarrow{\tilde f_*} \mathrm{Hom}\,(B_M, B_{M_2}^2) \xrightarrow{\tilde g_*} \mathrm{Hom}(B_M, B_{M_3}^3)$ is exact sequence, $\alpha \in \mathrm{Hom}\,(M, M_1)$ and $\widetilde{f\alpha}\in \ker \tilde g_*$. Since $\ker \tilde g_* = \operatorname{im} \tilde f_*$, then there is $\tilde h\in \mathrm{Hom}(B_M, B_{M_1}^1)$ such that $\widetilde{f\alpha}=\tilde f_* \tilde h$, so $\tilde f_* \tilde\alpha = \tilde f_* \tilde h$, but $\tilde f_*$ is injective, hence $\tilde\alpha = \tilde h$, i. e. $\tilde\alpha \in \mathrm{Hom}(B_M, B_{M_1}^1)$.

Sufficiency. Becarse $\tilde f$ is injective, it is easy to see that $\tilde f_*$ is injective, too. To see that $\operatorname{im} \tilde f_* = \ker \tilde g_*$, for any $\tilde\varphi \in \ker \tilde g_*$, we have $\tilde g_* \tilde\varphi = \tilde 0$, so $\widetilde{g\varphi}=\tilde 0$, hence $g\varphi=0$. Now there is $\alpha \in \mathrm{Hom}(M, M_1)$ such that $\varphi=f\alpha$, i. e. $\widetilde{f\alpha}=\tilde\varphi \in \ker \tilde g_*$, whence $\tilde\alpha \in \mathrm{Hom}(B_M, B_{M_1}^1)$, consequently $\tilde\varphi = \tilde f \tilde\alpha = \tilde f_* \tilde\alpha \in \operatorname{im} \tilde f_*$, i. e. $\ker \tilde g_* \leqslant \operatorname{im} \tilde f_*$. In addition to, for any $\tilde\psi \in \operatorname{im} \tilde f_*$, there is $\tilde\alpha \in \mathrm{Hom}(B_M, B_{M_1}^1)$ such that $\tilde f_* \tilde\alpha = \tilde\psi$, so $\tilde g_* (\tilde f_* \tilde\alpha) = \tilde g(\tilde f \tilde\alpha) = \widetilde{gf\alpha} = \tilde 0$, i. e. $\operatorname{im} \tilde f_* \leqslant \ker \tilde g_*$, hence

$\ker \tilde{g}_* = \mathrm{im}\ \tilde{f}_*$.

2)The proof is similar to the proof of 1).

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