The Initial Objects, Terminal Objects, Equalizers and Intersections on Categories of F_R^A-modules.

Zhao Jianli

Department of Mathematics, Liaocheng Teachers College, Shandong, 252059, P. R. China

Abstract: In this paper, we show that the category of F_R^A — modules is a top category, we obtain that the category of F_R^A-module has initial objects, terminal objects, equalizers and intersections.

Keywords: F_R^A-module, F — homomorphism, category of F_R^A-module, initial object, terminal object, top category, equalizer, intersection.

1 Introduction

[1], [2], [3] introduce the concepts of F_R^A — module and the category of F_R^A — modules, and discuss the properties of them. In this paper, we show that the category of F_R^A — modules is a top category, and we obtain that the category of F_R^A — module has initial objects, terminal objects, equalizers and intersections.

Let X be a nonempty set, L be a complete distributive lattice (with 0 and 1), A fuzzy subset A on X is characterised by a mapping $A: X \to L$. X^L denotes the set of whole fuzzy subset of X. In this paper R is a ring with identity $1 \neq 0$ and module which involved is an unitary left R — module.

Definition 1. 1. [1] Let M be a left R — module, A a fuzzy subring of R and $A(0)=1$, $B_M \in M^L$, if for all $x, y \in M$, $r \in R$, we have

1) $B_M(x-y) \geqslant B_M(x) \wedge B_M(y)$,

2) $B_M(0)=1$,

3) $B_M(rx) \geqslant A(r) \wedge B(x)$,

then B_M is called an F_R^A-submodule (or F_R^A-module).

Definition 1. 2. Let M and N be two R — modules, $f: M \longrightarrow N$ be an R — homomorphism,

B_M be an F_R^A—submodule of M, $\tilde{f}(B_M)$ is defined by

$$\tilde{f}(B_M)(y)=\begin{cases} \vee\{B_M(x)\mid x\in M, f(x)=y\}, & \text{if } f^{-1}(y)\neq\Phi, \\ 0, & \text{if } f^{-1}(y)=\Phi, \end{cases}$$

for all $y\in N$.

Definition 1. 3. Let M and N be two left R—module, $f: M\longrightarrow N$ be an R—homomorphism, B_M and C_N be F_R^A-submodule of M and N, respectively, if $\tilde{f}(B_M)\leqslant C_N$, then $\tilde{f}$ is called an F—homomorphism from B_M into C_N, writes $\tilde{f}: B_M\longrightarrow C_N$.

Definition 1. 4. The category of F_R^A-modules F_R^A—Mod is defined by:

1) Objects are all F_R^A-modules,

2) For all $B_M, C_N\in$ Obj (F_R^A—Mod), the set of morphisms is

$$\text{Hom}(B_M, C_N)=\{\tilde{f}\mid\tilde{f}\text{ is an arbitrary } F\text{—homomorphism from } B_M \text{ into } C_N\},$$

3) For all $\tilde{f}\in\text{Hom}(B_M, C_N)$, $\tilde{g}\in\text{Hom}(C_N, D_S)$, the composition of $\tilde{f}$ and $\tilde{g}$ is defined by $\tilde{f}\tilde{g}=\widetilde{fg}$.

Let R—Mod denotes the category of left R—modules.

Definition 1. 5. Let $B_M, C_N\in$ Obj(F_R^A—mod), if $N\subseteqq M$ and $B_M(x)\geqslant C_N(x)$, for all $x\in N$, then C_N is called the subobject of the object B_M.

2. Initial and terminal objects and top properties of F_R^A—modules

Let M be a left R—module, A be a fuzzy subring of R, two F_R^A—module of M are defined by

$$0_M: M\longrightarrow L,\qquad 0_M(m)=\begin{cases} 0, \text{if } m\neq 0, m\in M, \\ 1, \text{if } m=0, m\in M, \end{cases}$$

$$1_M: M\longrightarrow L,\qquad 1_M(m)=1, \forall\, m\in M.$$

Consequently, the diagram (1) and diagram (2) are admissible.

$$\begin{tikzcd} & \{0\} \arrow[dl] \arrow[dr, "0_M"] & \\ M \arrow[rr, "B_M"] & & L \end{tikzcd}$$

diagram (1)

$$\begin{tikzcd} & M \arrow[dl] \arrow[dr, "B_M"] & \\ \{0\} \arrow[rr, "1_{(0)}"] & & L \end{tikzcd}$$

diagram (2)

Hence the objects 0_M in category F_R^A—Mod is the initial objects of the category F_R^A—Mod and the objects $1_{\{0\}}$ is the terminal objects of the category F_R^A—Mod , therefore we have the following Proposition 2. 1.

Proposition 2. 1. The category F_R^A—Mod has initial objects and terminal objects.

By Proposition 2. 1, we have the following Proposition 2. 2.

Proposition 2. 2. The category F_R^A—Mod has zero objects.

Theorem 2. 3. The category F_R^A—Mod is additive category, but it is not abel category.

Proof. By [2], [3] and Proposition 2. 2, the category F_R^A—Mod has products, coproducts, kernels and cokernels and zero objects, hence the category F_R^A—Mod is additive category.

Let $\tilde{g}: C_N \longrightarrow B_M$ be a subobject of B_M, if $\tilde{g}$ is normal, then $C_N = \tilde{f}^{-1}(B_M)$. Hence for $M \neq 0$, $\tilde{1}: 0_M \longrightarrow 1_M$ is a subobjects of 1_M which is not a Kernel, therefore F_R^A-Mod is not an abel category.

Proposition 2. 4. Let M be a left R—module, the set $\Omega_L(M) = \{B_M \mid B_M \text{ is an } F_R^A\text{—module of } M\}$ is a complete lattice under the following order relation:

$$B_M \leqslant C_N \text{ iff } B_M(x) \leqslant C_M(x), \text{for all } x \in M.$$

Proof. Let $\{B_M^i \mid i \in I\} \subseteqq \Omega_L(M)$

$$(\bigwedge_{i\in I} B_M^i)(x) = \bigwedge_{i\in I} B_M^i(x),$$

$$\bigvee_{i\in I} B_M^i(x) = \bigwedge_{i\in I} \{C_M(x) \mid C_M \in \Omega_L(M), B_M^i \leqslant C_M, \text{ for all } i \in I\},$$

are the inf and sub of the collection $\{B_M^i \mid i \in I\}$, respectively.

Theorem 2. 5. The category F_R^A-Mod is a top category over R—Mod.

Proof. The proof is similar to Theorem 3. 4 of [4].

3 Equalizers and intersections

Theorem 3. 1 The category F_R^A-Mod has equalizers.

Proof. Let $B_M, C_N \in$ Obj (F_R^A-Mod), and $\tilde{f}_1, \tilde{f}_2: B_M \longrightarrow C_N$ be two morphisms in F_R^A-Mod. So we have two R-homomorphisms $f_1, f_2: M \longrightarrow N$ in R—Mod. But the category R-Mod has equalizers and let the equalizer of f, g in R-Mod be $i_k: K \longrightarrow M$, where

$$K=\{x\,|\,f_1(x)=f_2(x),x\in M\},$$

and i_k is the inclusion map. Eviently, K is an R-submodule of M. We define an F_R^A-module of K,

$$D_K:K\longrightarrow L, D_K(x)=B_M(x), \forall\ x\in K,$$

since

$$D_K(x)\leqslant B_M(i_k(x)), \text{for all } x\in K,$$

consequently, $\tilde{i}_k:D_K\longrightarrow B_M$ is an F—homomorphism. By the above construction we get that the following diagram holds in F_R^A-Mod.

$$\begin{tikzcd} D_K \arrow[r, "\tilde{i}_k"] & B_M \arrow[r, "\tilde{f}_1"] & H_N \arrow[d, equal] \\ H_N & B_M \arrow[l, "\tilde{f}_2"'] & D_K \arrow[l, "\tilde{i}_k"'] \end{tikzcd}$$

Let there is an F-homomorphism $g:C'_{K'}\longrightarrow B_M$ such that the following diagram holds in F_R^A-Mod.

$$\begin{tikzcd} D'_{K'} \arrow[r, "\tilde{g}"] & B_M \arrow[r, "\tilde{f}_1"] & C_N \arrow[d, equal] \\ C_N & B_M \arrow[l, "\tilde{f}_2"'] & D'_{K'} \arrow[l, "\tilde{g}"'] \end{tikzcd}$$

So

$$\begin{tikzcd} K' \arrow[r, "g"] & M \arrow[r, "f_1"] & N \arrow[d, equal] \\ N & M \arrow[l, "f_2"'] & K' \arrow[l, "g"'] \end{tikzcd}$$

holds in R-Mod. Since category R-Mod has equalizers, from the universal property there exists a unique R-homomorphism $k':K'\longrightarrow K$ in R—Mod such that $i_k k'=g$. Since we have the following diagram(3)

$$\begin{tikzcd} K' \arrow[rr, "D_{K'}"] \arrow[d, "k"'] \arrow[rrd, "g"] & & L \\ K \arrow[rru, "D_K"] \arrow[rr, "i_k"'] & & M \arrow[u, "B_M"'] \end{tikzcd}$$

diagram(3)

is communtive. Therefore

$$D'_{K'}(x)\leqslant B_M(x)\leqslant B_M(i_k k(x))$$
$$=B_M i_k(k(x))=D_k(x),$$

then $\tilde{k}'$ is an F—homomorphism. Consequently, the following diagram holds

$$\begin{tikzcd} D'_{K'} \arrow[r, "\tilde{k}"] & D_K \arrow[r, "\tilde{i}_k"] & B_M \arrow[d, equal] \\ & B_M & D'_{K'} \arrow[l, "\tilde{g}"'] \end{tikzcd}$$

Hence the pair $(D_K, \tilde{i}_k)$ is the equalizer of the pair of F—homomorphisms $\tilde{f}_1$ and $\tilde{f}_2$ in category F_R^A-Mod .

Theorem 3. 2. The category F_R^A—Mod has finite intersections.

Proof. Let$\{B'_{M_i} | i\in 1,2,\cdots n\}$ be the family of subobjects of the object B_M in F_R^A-Mod.

Let $M'=\bigcap_{i=1}^{n} M_i$, we define fuzzy subset $B'_{M'}$ of M',

$$B'_{M'}: M' \longrightarrow L$$

such that

$$B'_{M'}(x)=\wedge\{B'_{M_i}(x)\,|\, i=1,\cdots,n\}, \text{for all } x\in M',$$

it is easy to prove $B'_{M'}\in$ Obj(F_R^A-Mod).

For all $i, g_i: M_i \longrightarrow M$ are inclusion map, because

$$B'_{M_i}(x)\leqslant B_M(x)(f_i(x)), \text{for any } x\in M_i, i=1,\cdots,n$$

so for all $i=1,2,\cdots n, \tilde{g}_i: B'_{M_i}\longrightarrow B_M$ are F—homomorphism, let $\tilde{f}: D_H \longrightarrow B_M$ be an F—homomorphism which is foctored through each subobject B'_{M_i}, that is, for all $i\in I$, the following diagram holds.

$$\begin{tikzcd} D_H \arrow[r, "\tilde{f}_i"] & B'_{M_i} \arrow[r, "\tilde{g}_i"] & B_M \arrow[d, equal] \\ & B_M & D_H \arrow[l, "\tilde{f}"'] \end{tikzcd}$$

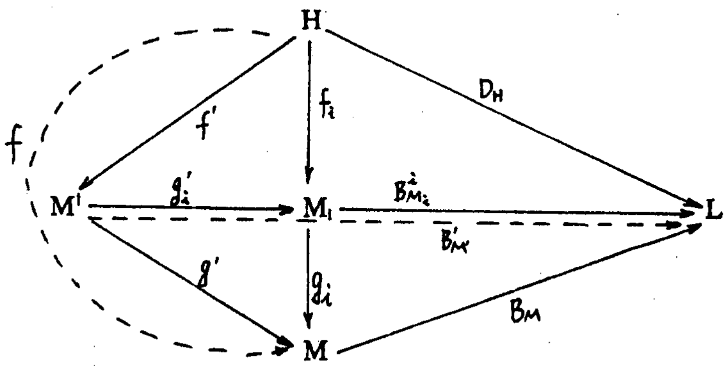
Hence

$$\begin{tikzcd} H \arrow[r, "f_i"] & M_i \arrow[r, "g_i"] & M \arrow[d, equal] \\ & M & H \arrow[l, "f"'] \end{tikzcd}$$

holds, for $i=1,2,\cdots n$. By the universal properties of intersection, there exists R—homomorphism $f': H \longrightarrow M'$ in R—Mod such that

$$\begin{tikzcd} H \arrow[r, "f'"] & M' \arrow[r, "g'"] & M \arrow[d, equal] \\ & M & H \arrow[l, "f"'] \end{tikzcd}$$

holds in R—Mod. Consider the following diagram(4)



diagram(4)

For all $x \in M$, we have

$$D_H(x) \leqslant B^i_{M_i}(f_i(x)) \qquad \forall\ i=1,2,\cdots,n,$$
$$g'_i f' = f_i,\ g_i g'_i = g' \qquad \forall\ i=1,2,\cdots n,$$

and $B^i_{M'}(x) = \wedge \{B^i_{M_i}(x) \mid i=1,2,\cdots,n\}$. Then $D_H(x) \leqslant B'_{M'}(f'(x))$, for all $x \in H$, so $\tilde{f}' : D_H \longrightarrow B'_{M'}$ is an F—homomorphism and

$$\begin{tikzcd} D_H \arrow[r, "\tilde{f}'"] & B'_{M'} \arrow[r, "\tilde{g}'"] & B_M \arrow[d, equal] \\ & B_M & D_H \arrow[l, "\tilde{f}"'] \end{tikzcd}$$

holds in $F^{\wedge}_R$-Mod. Therefore suboject $B^i_{M'}$ together with the family of F—homomorphisms

$$\{\tilde{g}^i_i = B'_{M'} \longrightarrow B^i_{M_i} \mid i=1,2,\cdots,n\}$$

is the intersection of the family of subobjects $\{B^i_{M_i} \mid i=1,\cdots,n\}$ in $F^{\wedge}_R$-Mod.

Reference

[1] Zhao Jianli, Shi Kaiquan, Yu Mingshan, Fuzzy Modules over Fuzzy Rings, The J. Fuzzy Math, 3(1993), 531—539.

[2] Zhao Jianli, $F^{\wedge}_R$-modules and $F^{\wedge}_R$-modules Categories The J. Fuzzy Math(to appear).

[3]Zhao Jianli, Fuzzy homological theroy in $F^{\wedge}_R$-modules(I), The J. Fuzzy Math (to appear).

[4] S. R. LOPEZ-PERMOUTH and D. S. MALIK, On Categories of Fuzzy Modules, Inform. Sci, (52)1990, 211-220.

[5] S. M. A. Zaidi and Q. A. Ansari, Some results on categories of L-fuzzy subgroups, Fuzzy sets and systems 64(1994) 249-256.

[6] T. S. Blyth, Categories (Longman, London, 1987).

[7] B. Mitchell, Theory of Categories (Academic Press, New York, 1965).

[8] O. Wyler, On the categories of general topology and topological algebra, Arch Math (Basel) 22:7—17(1971).