Pointwise Charactrization of Support-Preserving Fuzzy Order Homomorphism*

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Abstract: In this paper, the concept of pointwise fuzzy mapping is introduced. Using this concept we give a pointwise charactrization of the support-preserving fuzzy order homomorphism

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Throughout this paper, L, L_1 and L_2 always denote completely distributive lattices, i.e. molecular lattices. 0 and 1 are their smallest element and greatest element, respectively. Let X and Y be non-empty crisp sets. For $A \in L^X$, write $supp\ A = \{x \mid x \in X, A(x) > 0\}$ and $hgt\ A = \bigvee\{A(x) \mid x \in X\}$, they are called the support and height of A, respectively. We denote the set of all L-fuzzy points in X by $\tilde{X}(L)$. A L-fuzzy point with support point x and height λ is denote by x_λ. We assume that for the empty family $\emptyset$, $\bigvee \emptyset = 0$.

Definition 1[1,2] A mapping $f : L_1 \to L_2$ is called generalized order homomorphism if (1) $f(x) = 0$ iff $x = 0$; (2) f and f^{-1} are union-preserving, where

$$f^{-1}(y) = \bigvee\{x \in L_1 \mid f(x) \le y\}.$$

The mapping $F : L_1^X \to L_2^Y$ is called a fuzzy order homomorphism if F is a generalized order homomorphism.

Definition 2[3] A fuzzy order homomorphism $F : L_1^X \to L_2^Y$ is said to be support-preserving if it maps L_1-fuzzy sets having same support on X to L_2-fuzzy sets having same support on Y.

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Lemma 1 *Let $A_t \in L^X (t \in T)$. Then*

$$supp \bigvee\{A_t \mid t \in T\} = \bigvee\{supp\ A_t \mid t \in T\}.$$

The proof is straightforward, and is omitted.

Theorem 1 *Let $F : L_1^X \to L_2^Y$ be a fuzzy order homomorphism. Then F is support-preserving if and only if*

$$supp\ F(x_\lambda) = supp\ F(x_\mu), \quad \text{for all } \lambda,\ \mu \in L_1 - \{0\},\ x \in X.$$

Proof The necessity is obvious.

Sufficiency. Since F is a fuzzy order homomorphism, for any $A, B \in L_1^X$ with $supp\ A = supp\ B$,

$$\begin{aligned} F(A) &= F(\bigvee\{x_{A(x)} \mid x \in supp\ A\}) = \bigvee\{F(x_{A(x)}) \mid x \in supp\ A\}, \\ F(B) &= \bigvee\{F(x_{B(x)}) \mid x \in supp\ B\}. \end{aligned}$$

Hence, by Lemma 1 we have

$$\begin{aligned} supp\ F(A) &= \bigvee\{supp\ F(x_{A(x)}) \mid x \in supp\ A\} \\ &= \bigvee\{supp\ F(x_{B(x)}) \mid x \in supp\ B\} = supp\ F(B), \end{aligned}$$

and so F is support-proserving.

Corollary 1 *Let L_1 be* reqular[2], *Then any fuzzy order homomorphism $F : L_1^X \to L_2^Y$ is support-preserving.*

Proof It can be proved by Proposition 2 in [2] and Theorem 1.

Definition 3 A mapping $\tilde{f} : \tilde{X}(L_1) \to \tilde{Y}(L_2)$ is called a pointwise fuzzy mapping if the following conditions are satisfied:

(i) $\tilde{f}(x_{\vee\lambda_t}) = \vee \tilde{f}(x_{\lambda_t})$, for any $\lambda_t \in L_1 - \{0\}$ $(t \in T)$;

(ii) $\tilde{f}^{-1}(y_{\vee\mu_t}) = \vee \tilde{f}^{-1}(y_{\mu_t})$, for any $\mu_t \in L_2 - \{0\}$ $(t \in T)$,

Where $\tilde{f}^{-1}(b) = \vee\{a \in \tilde{X}(L_1) \mid \tilde{f}(a) \le b\}$.

Remark 1 The above concept is different from the concept of pointwise fuzzy mapping introduced in [5].

Lemma 2 *Let $\tilde{f} : \tilde{X}(L_1) \to \tilde{Y}(L_2)$ be a pointwise fuzzy mapping. Then*

$$supp\ \tilde{f}(x_\lambda) = supp\ \tilde{f}(x_\mu), \quad \text{for } \lambda,\ \mu \in L_1 - \{0\} \text{ and } x \in X.$$

Proof By the definition of pointwise fuzzy mapping,

$$\tilde{f}(x_{\lambda\vee\mu}) = \tilde{f}(x_\lambda)\bigvee\tilde{f}(x_\mu).$$

Noting $\tilde{f}(x_{\lambda\vee\mu}) \in \tilde{Y}(L_2)$, and so $supp\ \tilde{f}(x_\lambda) = supp\ \tilde{f}(x_\mu)$.

Lemma 3 *Let $\tilde{f} : \tilde{X}(L_1) \to \tilde{Y}(L_2)$ be a pointwise fuzzy mapping. If there exist an ordinary mapping $f : X \to Y$ and a family of mappings $\{\varphi_x \mid \varphi_x : L_1 \to L_2, x \in X\}$ such that $\tilde{f}(x_\lambda) = [f(x)]_{\varphi_x(\lambda)}$ for all $x_\lambda \in \tilde{X}(L_1)$, then for $y_\mu \in \tilde{Y}(L_2)$*

$$\tilde{f}^{-1}(y_\mu)(x) = \begin{cases} {\varphi_x}^{-1}(\mu), & \text{if } f(x) = y, \\ 0, & \text{if } f(x) \neq y. \end{cases} \tag{1}$$

Proof Since $\tilde{f}^{-1}(y_\mu) = \bigvee\{x_\lambda \mid \tilde{f}(x_\lambda) \le y_\mu\} = \bigvee\{x_\lambda \mid [f(x)]_{\varphi_x(\lambda)} \le y_\mu\}$, we have

$$\tilde{f}^{-1}(y_\mu) = \begin{cases} \bigvee\{\lambda \in L_1 \mid \varphi_x(\lambda) \le \mu\}, & \text{if } f(x) = y, \\ 0, & \text{if } f(x) \neq y, \end{cases}$$

i.e. (1) holds.

Theorem 2 *A mapping $\tilde{f} : \tilde{X}(L_1) \to \tilde{Y}(L_2)$ is a pointwise fuzzy mapping if and only if there exist an ordinary mapping $f : X \to Y$ and a family of generalized order homomorphisms $\{\varphi_x \mid \varphi_x : L_1 \to L_2, x \in X\}$ such that*

$$\tilde{f}(x_\lambda) = [f(x)]_{\varphi_x(\lambda)}, \quad \text{for } x_\lambda \in \tilde{X}(L_1). \tag{2}$$

Proof Suppose that $\tilde{f} : \tilde{X}(L_1) \to \tilde{Y}(L_2)$ is a pointwise fuzzy mapping. Define the mappings $f : X \to Y$ and $\varphi_x : L_1 \to L_2$ (for every $x \in X$) as follows, respectively:

$$f(x) = supp\ \tilde{f}(x_1), \tag{3}$$

$$\varphi_x(\lambda) = hgt\ \tilde{f}(x_\lambda), \text{ for } \lambda \neq 0 \text{ and } \varphi_x(0) = 0. \tag{4}$$

According to Lemma 1, $supp\ \tilde{f}(x_\lambda) = supp\ \tilde{f}(x_1) = f(x)$, hence (2) holds. It remains only to show that $\varphi_x : L_1 \to L_2$ is a generalized order homomorphism.

Obviously, $\varphi_x(\lambda) = 0$ if and pnly if $\lambda = 0$. By (2), (4) and Definition 3, we have

$$\begin{aligned} \varphi_x(\bigvee\lambda_t) &= hgt\ \tilde{f}(x_{\vee\lambda_t}) = hgt\ (\bigvee\tilde{f}(x_{\lambda_t})) \\ &= hgt\ (\bigvee[f(x)]_{\varphi_x(\lambda_t)}) = hgt\ [f(x)]_{\vee\varphi_x(\lambda_t)} = \bigvee\varphi_x(\lambda_t). \end{aligned}$$

For $x \in X$, write $f(x) = y$. By (2), Lemma 3 and Definition 3, we have

$$\begin{aligned} {\varphi_x}^{-1}(\bigvee\mu_t) &= \tilde{f}^{-1}(y_{\vee\mu_t})(x) = [\bigvee\tilde{f}^{-1}(y_{\mu_t})](x) \\ &= \bigvee[\tilde{f}^{-1}(y_{\mu_t})](x) = \bigvee{\varphi_x}^{-1}(\mu_t). \end{aligned}$$

This shows that φ_x is a generalized order homomorphism.

Conversely, let $\tilde{f}:\tilde{X}(L_1)\to\tilde{Y}(L_2)$ be a mapping defined by (2), where $f:X\to Y$ is a mapping and $\{\varphi_x \mid \varphi_x: L_1\to L_2, x\in X\}$ is a family of generalized order homomorphisms. Then it is easy to verify that $\tilde{f}$ satisfies the conditions (i) and (ii) of Definition 3.

(i) $\tilde{f}(x_{\vee\lambda_t})=[f(x)]_{\varphi_x(\vee\lambda_t)}=[f(x)]_{\vee\varphi_x(\lambda_t)}=\bigvee[f(x)]_{\varphi_x(\lambda_t)}=\bigvee\tilde{f}(x_{\lambda_t})$.

(ii) When $y\neq f(x)$, $\tilde{f}^{-1}(y_{\vee\mu_t})(x)=0=[\bigvee\tilde{f}^{-1}(y_{\mu_t})](x)$; When $y=f(x)$, we have

$$\begin{aligned}\tilde{f}^{-1}(y_{\vee\mu_t})(x) &= {\varphi_x}^{-1}(\bigvee\mu_t)=\bigvee{\varphi_x}^{-1}(\mu_t)=\bigvee\tilde{f}^{-1}(y_{\mu_t})(x)\\ &= [\bigvee\tilde{f}^{-1}(y_{\mu_t})](x).\end{aligned}$$

Hence $\tilde{f}^{-1}(y_{\vee\mu_t})=\bigvee\tilde{f}^{-1}(y_{\mu_t})$.
Therefore $\tilde{f}$ is a pointwise fuzzy mapping. This completes the proof.

In what follows, we discuss the relations between pointwise fuzzy mapping and support-preserving fuzzy order homomorphism.

Lemma 4[4] *A mapping $F: L_1^X\to L_2^Y$ is a support-preserving fuzzy order homomorphism if and only if there exist an ordinary mapping $f:X\to Y$ and a family of generalized order homomorphisms $\{\varphi_x\mid\varphi_x:L_1\to L_2, x\in X\}$ such that F is the malti-induced mapping of f and $\{\varphi_x \mid x\in X\}$.*

Theorem 3 *Suppose that $F: L_1^X\to L_2^Y$ is a support-preserving fuzzy order homomorphism. Then the restriction $\tilde{f}=F\mid_{\tilde{X}(L_1)}$ of F on $\tilde{X}(L_1)$ is a pointwise fuzzy mapping. Conversely, suppose that $\tilde{f}:\tilde{X}(L_1)\to\tilde{Y}(L_2)$ is a pointwise fuzzy mapping. Then there exist a unique support-preserving fuzzy order homomorphism $F: L_1^X\to L_2^Y$ such that $\tilde{f}=F\mid_{\tilde{X}(L_1)}$.*

Proof Suppose that F is a support-preserving fuzzy order homomorphism, by Lemma 4, there exist an ordinary mapping $f:X\to Y$ and a family of generalized order homomorphisms $\{\varphi_x\mid\varphi_x:L_1\to L_2, x\in X\}$ such that F is the multi-induced mapping of f and $\{\varphi_x\mid x\in X\}$, i.e.

$$F(A)(y)=\bigvee\{\varphi_X(A(x))\mid f(x)=y\},\quad \text{for } A\in L_1^X \text{ and } y\in Y.$$

From this it follows that

$$F(x_\lambda)(y)=\begin{cases}\varphi_x(\lambda), & f(x)=y,\\ 0, & f(x)\neq y.\end{cases}\tag{5}$$

It is not difficult to varify that $\varphi_x(\lambda)>0$ for all $\lambda\in L_1-\{0\}$. In fact, if there exists λ_0 such that $\varphi_x(\lambda_0)=0$, then

$${\varphi_x}^{-1}(0)=\bigvee\{\lambda\in L_1\mid\varphi_x(\lambda)\le 0\}\ge\lambda_0>0.$$

This is a contradiction, since φ_x is a generalized order homomorphism. So by (5) we have $F(x_\lambda) = [f(x)]_{\varphi_x(\lambda)}$. Hence from Theorem 2 we know that $F|_{\tilde{X}(L_1)}$ is a pointwise fuzzy mapping.

Conversely, suppose that $\tilde{f}$ is a pointwise fuzzy mapping. By Theorem 2 there exist an ordinary mapping $f: X \to Y$ and a family of generalized order homomorphisms $\{\varphi_x \mid \varphi_x : L_1 \to L_2, x \in X\}$ such that $\tilde{f}(x_\lambda) = [f(x)]_{\varphi_x(\lambda)}$ for $x_\lambda \in \tilde{X}(L_1)$. We define a mapping $F : L_1^X \to L_2^Y$ as follows:

$$F(A)(y) = \bigvee\{\varphi_x(A(x)) \mid f(x) = y\}, \quad \text{for } A \in L_1^X,\ y \in Y.$$

namely, F is a multi-induced mapping of f and $\{\varphi_x \mid x \in X\}$. From this it follows that $F(x_\lambda) = [f(x)]_{\varphi_x(\lambda)}$, i.e. $F|_{\tilde{X}(L_1)} = \tilde{f}$. From Lemma 4 we know that F is a support-preserving fuzzy order homomorphism.

To prove the uniqueness, assume that $G : L_1^X \to L_2^Y$ is a support- preserving fuzzy order homomorphism and $G|_{\tilde{X}(L_1)} = \tilde{f}$. For any $A \in L_1^X$, we have

$$\begin{aligned} G(A) &= G(\bigvee\{x_{A(x)} \mid x \in suup\ A\}) = \bigvee\{G(x_{A(x)}) \mid x \in supp\ A\} \\ &= \bigvee\{\tilde{f}(x_{A(x)}) \mid x \in supp\ A\} = \bigvee\{[f(x)]_{\varphi_x(A(x))} \mid x \in supp\ A\}. \end{aligned}$$

Thus $G(A)(y) = F(A)(y)$, for all $y \in Y$, and so $G = F$. This completes the proof.

Remark 2 From Theorem 3, we know that a support-preserving fuzzy order homomorphism $F : L_1^X \to L_2^Y$ can be defined by a corresponding pointwise fuzzy mapping $\tilde{f} : \tilde{X}(L_1) \to \tilde{Y}(L_2)$, uniquely.

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