Some results of fuzzy modules over fuzzy rings

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Abstract: In this paper, we give some characteristic description and a kind of representation of the fuzzy modules over fuzzy rings. The paper also discusses some important properties of fuzzy homomorphism.

Key Words: Fuzzy subring, FF- module, fuzzy homomorphism.

1 . Introduction

The concepts of fuzzy sets was introduced by Zedeh [9].This concepts was applied to the theory of module by Negoita and Ralescu in [3].Since then many authors further study the properties of fuzzy module in [4-8].But these papers did not establish the connections between fuzzy modules and fuzzy rings,then we can not study fuzzy rings from outside. The paper [1], [2] introduced the concepts of fuzzy modules over fuzzy ring and its categories, respectively, these concepts establish the connections between fuzzy modules and fuzzy rings.Now the [1], [2] provied the basic knowledge reaching fuzzy ring from outside.In this paper, we will study some characteristic description and some important properties of fuzzy module over fuzzy ring.

2 . Some Characteristic deseriptions of FF- module

Let X be any nonempty set, and L a complete distributive lattice with 0 and 1. AFuzzy subset μ_X on X is characterised by a mapping $\mu_X : X \to L$, $F(X)$ denotes the set of whole fuzzy subset of X. In this paper, R is a ring with identity $1 \neq 0$ and module which involved is an unitary R-module.

Definition 2.1 Let (M, R, μ_R, μ_M) be quarternary groups, where M is a left R-module, μ_R is a fuzzy subring of ring R, $\mu_M \in F(M)$, if for all $x, y \in M$, $r \in R$, We have

$1)\ \mu_M(x+y) \geq \mu_M(x) \wedge \mu_M(y)$,

$2)\ \mu_M(x) \geq \mu_M(-x)$,

$3)\ \mu_M(0)=1$,

$4)\ \mu_M(rx) \geq \mu_R(r) \vee \mu_M(x)$

then (M, R, μ_R, μ_M) is called a fuzzy submodule of M over fuzzy subring μ_R. In brief μ_M is a μ_R-*FF* module or *FF*- module.

In[1], [2], the definition of fuzzy module over fuzzy ring is not same.In this paper, the way of definition 2.1 is same with way of [1].In fact, if we changed 4) into

$4)'\ \mu_M(rx) \geq \mu_M(r) \wedge \mu_R(x)$

then the way of definition 2.1 is a same with way of [2]. According to 4), we have

$$\mu_M(rx) \geq \mu_M(x),$$

so μ_R-*FF* module is always *FF*- module.

Theorem 2.2 *Let μ_R be Fuzzy subring of ring R, if there exists $a \in R$ such that $\mu_R(a)=1$, then μ_R is a μ_R-FF module iff μ_R is a fuzzy idea of R.*

Proof. Necessity is trivial.To see the sufficiency, we only prove $\mu_R(0)=1$.In fact

$$\mu_R(0)=\mu_R(a-a) \geq \mu_R(a) \wedge \mu_R(a)=1$$

Theorem 2.3 *Let μ_R be a fuzzy subring of R, M be a left R-module, $\mu_M \in F(M)$, then μ_M is a μ_R-FF module iff*

1) there exists $x \in M$ such that $\mu_R(x)=1$,

2) for all $\alpha \in L$, $(\mu_R)_\alpha$ is a left R-module,

3) for all $x \in (\mu_M)_\alpha$ and $r \in (\mu_R)_\beta$, there exists $\gamma \geq \alpha \vee \beta$ such that $rx \in (\mu_M)_\gamma$.

Proof. " $\Longrightarrow$ " 1) is trivial,

2) for all $x, y \in (\mu_M)_\alpha$, $a, b \in R$, then

$$\mu_M(ax+by) \geq \mu_M(ax) \wedge \mu(by)$$
$$\geq (\mu_R(a) \vee \mu_M(x)) \wedge (\mu_R(b) \vee \mu_M(y)) \geq \mu_M(x) \wedge \mu_M(y)) \geq \alpha$$

that is, $ax+by \in (\mu_M)_\alpha$. Consequently, $(\mu_M)_\alpha$ is a left R-module.

3) Let $\gamma=\mu_M(rx)$, then $rx \in (\mu_M)_\gamma$, so $\gamma=\mu_M(rx) \geq \mu_R(r) \vee \mu_M(x) \geq \alpha \vee \beta$.

" $\Leftarrow$ " For any $x, y \in M, r \in R$, let $\mu_M(x) \wedge \mu_M(y)=\alpha$, then $\exists\ \gamma \geq \mu_M(x) \vee \mu_R(r)$ such that $rx \in (\mu_M)_\gamma$ hence.

$\mu_M(x+y) \geq \alpha \geq \mu_M(x) \wedge \mu_M(y)$,

$\mu_M(rx) \geq \gamma \geq \mu_M(x) \vee \mu_M(a)$.

We can easily prove $\mu_M(0)=1$, $\mu_M(x) \geq \mu_M(-x)$. Consequently, μ_M is a μ_R-*FF* module.

Theorem 2.4 *Let μ_R and μ_M be fuzzy subset of a ring R and a module M, respectively, then (M, R, μ_R, μ_M) is a μ_R-FF module iff $\bigvee_{x \in M} \mu_M(x)=1$, and there exists a submodule family of M*

$$A = \{X_\alpha / \alpha \in L\}$$

and a subring family

$$B = \{Y_\alpha / \alpha \in L\}$$

such that

1) $X_\alpha \cap X_\beta = X_{\alpha \vee \beta}$, $Y_\alpha \cap Y_\beta = Y_{\alpha \vee \beta}$

2) for any $H \subseteq L$, $\bigcap_{\alpha \in H} Y_\alpha \subseteq Y_{\bigvee_{\alpha \in H} \alpha}$, $\bigcap_{\alpha \in H} X_\alpha \subseteq X_{\bigvee_{\alpha \in H} \alpha}$,

3) if $x \in X_\beta$, $a \in Y_\alpha$, *there exists* $\gamma \geq \alpha \vee \beta$ *such that* $ax \in X_\gamma$,

4) X_α *is a left R-module,*

5) $\mu_R = \bigcup_{\alpha \in L} \alpha \tilde{Y}_\alpha$, $\mu_M = \bigcup_{\alpha \in L} \alpha \tilde{X}_\alpha$, *here* $\tilde{Y}_\alpha$ *and* $\tilde{X}_\alpha$ *indicate characteristic function of* μ_R *and* μ_M, *respectively.*

Proof. " $\Rightarrow$ " The result follows by theorem 1.3 and the decomposed theorem of fuzzy subset.

" $\Leftarrow$ " We first prove that the μ_R is a fuzzy subring of R. For all $a \in R$, *if* $a \notin Y_\alpha$ (for any $\alpha \in L$), by 5) we have $\mu_R(a)=0$, if there exists $\alpha \in L$ such that $a \in Y_\alpha$, then

$\mu_R(a)=\bigvee_{a\in Y\alpha}\alpha$. thus when we suppose that supremum of empty set is 0 , we have

$\mu_R(a)=\bigvee_{a\in Y\alpha}\alpha$, For any a , $b\in R$, let $\mu_R(a)=\bigvee_{a\in Y\alpha}\alpha=\lambda$, $\mu_R(b)=\bigvee_{b\in Y\beta}\beta=\mu$, because

$a\in\bigcap_{a\in Y\alpha}Y\alpha\subseteq Y_{\vee_{a\in Y\alpha}\alpha}=Y\lambda$, $b\in\bigcap_{b\in Y\beta}Y\beta\subseteq Y_{\vee_{b\in Y\beta}\beta}=Y\mu$, but $Y_\lambda\cap Y_{\lambda\wedge\mu}=Y_\lambda$,

$Y_\mu\cap Y_{\lambda\wedge\mu}=Y_\mu$, so $Y_{\lambda\wedge\mu}\supseteq Y_\lambda$, $Y_{\lambda\wedge\mu}\supseteq Y_\mu$, hence $a-b\in Y_{\lambda\wedge\mu}$ *i . e .*

$$\mu_R(a\text{-}b)=\bigvee_{\alpha\in L}\left(\alpha\wedge\tilde{Y}_\alpha(a-b)\right)\geq(\lambda\wedge\mu)\wedge 1$$

$$=\lambda\wedge\mu=\mu_R(a)\wedge\mu_R(b)$$

Similarly , $\mu_R(ab)\geq\mu_R(a)\wedge\mu_R(b)$, so μ_R is fuzzy subring of R .

For any $a\in R$ and $x\in M$, let $\mu_R(a)=\alpha$, $\mu_M(x)=\beta$, then there exists $\gamma\geq\alpha\vee\beta$ such that $ax\in X_\gamma$, thus

$$\mu_M(ax)=\vee\ (\alpha\wedge X_\alpha\ (ax))\geq\gamma\geq\alpha\vee\beta=\mu_R(a)\vee\mu_M(x)$$

We can easily prove $\mu_M(x+y)\geq\mu_M(x)\wedge\mu_M(y)$, $\mu_M(-x)\geq\mu_M(x)$, $\mu_M(0)=1$, for all $x,y\in M$, Consequently , μ_M is a μ_R-*FF* module .

Definition 2 . 5 Let $\mu_X\in F(X)$, for all $x\in X$, let

$$Ker\ x=\{\alpha\in L|x\in(\mu_X)_\alpha\}$$

then $Ker\ x$ is called kernel of X in L .

Propoition 2 . 6 *Let $\mu_X\in F(X)$, $x\in X$, then $Ker\ x=\{a\in L|x\in(\mu_X)_\alpha\}$, and $Ker\ x$ is an ideal of L which is generated by X in L , and $\mu_X(x)=\sup Ker\ x$.*

Theorem 2 . 7 *Let μ_R and μ_M be fuzzy sets of R and M , respectively , M be a left R-module , then μ_M is a μ_R-FF module iff .*

1) $Ker\ (x+y)\supseteq Ker\ x\cap Ker\ y$, $Ker\ (a+b)\supseteq Ker\ a\cap Ker\ b$,

2) $Ker\ (-x)=Ker\ x$, $Ker\ (-a)=Ker\ a$,

3) $Ker\ (ab)\supseteq Ker\ a\cap Ker\ b$, $Ker\ ax\supseteq Ker\ a\cup Ker\ x$,

4) $\bigvee_{x\in M}\mu_M(x)=1$

for all x , $y\in M$, a , $b\in R$.

Proof . The proof is easy .

Definition 2 . 8 Let M be a left R-module, μ_R be a fuzzy subring of R, A, $B \in F(M)$, $r \in R$, the fuzzy subsets of M: $A \cap B$, $A+B$, rA, $-A$ are defined by :

$$(A \cap B)(x)=A(x) \wedge B(x),$$

$$(A+B)(x)=\bigvee_{x=x_1+x_2} [A(x_1) \wedge B(x_2)],$$

$$(rA)(x)=\bigvee_{rx_1=x} [\mu_R(r) \vee A(x_1)],$$

$$(-A)(x)=A(-x)$$

for all $x \in M$.

Theorem 2 . 9 *Let M be a left R-module, μ_R be a fuzzy subring of R, $\mu_M \in F(M)$, then μ_M is a μ_R-FF module iff .*

1) $\bigvee_{x \in M} \mu_M(x)=1$,

2) $\mu_M+\mu_M \subseteq \mu_M$, $-\mu_M \subseteq \mu_M$, $r\mu_M \subseteq \mu_M$

for all $r \in R$.

Proof . The proof is easy .

3 、 Fuzzy Homomorphism

Definition 3 . 1 Let M and N be two R-module , $f: M \to N$ be an R-homomorphism, μ_M and η_N be μ_R-FF modules of M and N, respectively ,

(1) $\tilde{f}(\mu_M)(y)= \vee\{\mu_M(x) | x \in M, f(x)=y\}$, for all $y \in f(M)$,

(2) if $\tilde{f}(\mu_M) < \eta_N$, then $\tilde{f}$ is called a fuzzy homomorphism from μ_M into η_N, we write : $\tilde{f}: \mu_M \to \eta_N$,

(3) if $\tilde{f}(\mu_M)=\eta_N$, then $\tilde{f}$ is called an F-isomorphism from μ_M into η_N , we

write : $\mu_M \cong \eta_N$.

Proposition 3 . 2[1] *Let μ_M , η_N be μ_R -FF module of M and N , respectively , then $\mu_M \cong \eta_N$ iff $\exists$ $f: M \rightarrow N$ is an R-isomorphism such that $\mu_M f(m)= \eta_N (m)$, for all $m \in M$.*

Theorem 3 . 3 (the extension theorem of isomorphism) *Let $f: {}_RM \rightarrow {}_RM'$ is an isomorphism of R-module , μ_M is a μ_R -FF module of ${}_RM$, then there exists a μ_R -FF module $\eta_{M'}$ of ${}_RM'$ such that $\mu_M \cong \eta_{M'}$.*

Proof . For any $x' \in {}_RM'$, because f is an isomorphism map , then there is a unique $x \in M$, such that $f(x)=x'$. Define :

$\eta_{M'}: M' \rightarrow L$, $x' \rightarrow \mu_M (x)$, where $f(x)=x'$,

then $\eta_{M'}$ is a μ_R -FF module . In fact , for all x' , $y' \in M'$, there are x , $y \in M$ such that $f(x)=x'$, $f(y)=y'$, thus $f(x+y)=x'+y'$, $f(-x)=-x'$, $f(rx)=rx'$, hence

$\eta_{M'} (x'+y')=\mu_M (x+y) \geq \mu_M (x) \wedge \mu_M (y) = \eta_{M'} (x') \wedge \eta_{M'} (y')$,

$\eta_{M'} (-x')=\mu_M (-x)=\mu_M (x)= \eta_{M'} (x')$,

$\eta_{M'} (0)=\mu_M (0)=1$,

$\eta_{M'} (rx')=\mu_M (rx) \geq \mu_R (r) \vee \mu_M (x)=\mu_R (r) \vee \eta_M (x')$,

that is , $\eta_{M'}$ is a μ_R -FF module of ${}_R$M'.

According to the define of $\eta_{M'}$, we have

$$\eta_{M'} (f(x))= \eta_{M'} (x')=\mu_M (x) ,$$

by the proposition 3 . 2 , we have $\mu_M \cong \eta_{M'}$.

Theorem 3 . 4 *Let (M , R ,μ_R ,μ_M) is a μ_R-FF module , then there exists an FF- module (H,R, η_R ,η_H) of $Hom_R(R , M)$ such that $\mu_M \cong \eta_H$.*

Proof let $H=Hom_R (R , M)$, then H is a left R-module , and there exists an isomorphism of left R-module

$$\rho : {}_RM \rightarrow Hom_R (R,M) , \rho(x)a=ax , a \in R ,$$

let $\eta_R = \mu_R$, for any $f \in Hom_R (R , M)$, there exists a unique $x \in M$ such that $\rho(x)=f$, let

$\eta_H(f)=\mu_M(x)$, by the extension theorem of isomorphism we see that $\widetilde{\rho}: \mu_M \to \eta_H$ is an isomorphism of μ_R-*FF* module, i. e. $\mu_M \cong \eta_H$.

let $(Hom_R(R, M), R, \mu_R, \eta_H)=Hom_R(R, \mu_M)$, then $\mu_M \cong Hom_R(R, \mu_M)$.

Definition 3 . 5 Let ${}_RM_S$ be (R, S)-bimodule, (M, R, μ_R, μ_M) and (M, S, μ_S, μ_M) be left μ_R-*FF* module and right μ_S-*FF* module, respectively, then$(M, R, S, \mu_R, \mu_S, \mu_M)$ is called (μ_R, μ_S)-*FF* bimodule .In brief μ_M is a (μ_R, μ_S) -*FF* bimodule or *FF*- bimodule .

Theorem 3 . 6 *Let ${}_RM_S$ be (R,S)-bimodule, e is a non-zero idempotent element in ring R, then for any (μ_{eRe}, μ_S)-FF bimodule μ_{eM} of eM, there exists (μ_{eRe}, μ_S)-FF bimodule η_H of $Hom_R(Re, M)$ such that $\mu_M \cong \eta_H$.*

Proof . By the Proposition 4 . 6 in[10], we see that .

$$\rho: eM \to Hom_R(Re, M),\ \rho(ex)re=rem$$

is an isomorphism of (eRe, S) - bimodule . Let $\eta_{eR} = \mu_{eR}$, $\eta_S = \mu_S$, then for any $f \in Hom_R(Re, M)=H$. there exists a unique $ex \in eM$ such that $\rho(ex)=f$. Let

$$\eta_H(f)=\mu_{eM}(ex)$$

then $(H, eRe, S, \eta_{eRe}, \eta_S, \eta_H)$ is an (η_{eRe}, η_S)-FF bimodule,and $\eta_H \cong \mu_{eM}$.

Theorem 3.7 *Let M be (R,S)—bimodule, f be a non-zero idemptent element in ring S, then for any (μ_R, μ_{fSf})-FF bimodule μ_{Mf}, there exists (μ_R, μ_{fSf})-FF bimodule η_H such that $\mu_{Mf} \cong \eta_H$.*

Proof . The proof is similar to the proof of Theorem 3.6.

Let $\widetilde{\phi}: \eta_R \to \mu_S$ be a homomorphism of fuzzy ring . For all μ_S-*FF* module , we can induce an η_R-*FF* module . In fact, $\phi: R \to S$ is a homomorphism of ring, if (M, S, η_S, η_M) is an *FF*- module, let $rm=\phi(r)m$, then M is a R-module, let

$$\eta_M(m)=\mu_M(m), \quad m \in M.$$

It is evident that (M, R, η_R, η_M) satisfies the conditions 1), 2), 3)in Definition 2 . 1 .In addition to, for any $r \in R$, $x \in M$, we have

$$\eta_M(rx)=\eta_M(\phi(r)x)=\mu_M(\phi(r)x)\geqslant\mu_S(\phi(r))\vee\mu_M(x)\geqslant\eta_R(r)\vee\eta_M(x),$$

hence η_M is an η_R-*FF* module. Certainly, μ_M is also an η_R-*FF* module. For η_S-*FF* homomorphism $f:\mu_M\to\mu_N$, since

$$f(rm)=f(\phi(r)m)=\phi(r)f(m)=rf(m)$$

so f is an R-homomorphism. Thus $(\eta_N f)(m)=\mu_N(f(m))\geqslant\mu_M(m)=\eta_M(m)$. Consequently, $\tilde{f}:\mu_M\to\mu_N$ is a μ_R-*FF* homomorphism, In a word the following theorem 3.9 is right.

Theorem 3.9 *Let* $\tilde{\phi}:\eta_R\to\mu_S$ *be a homomorphism of fuzzy ring, then*

$$T_{\tilde{\phi}}:\mu_S\text{-}FF\ mod\to\eta_R\text{-}FF\ mod$$

$$(M,S,\mu_S,\mu_M)\to(M,R,\eta_R,\eta_M),$$

$$\tilde{f}:\mu_M\to\mu_N\to\tilde{f}:\eta_M\to\eta_N$$

is a contravarian functor, where $\eta_M=\mu_M$.

The $T_{\tilde{\phi}}$ is called a functor of change of ring. If (M,R,μ_R,μ_M) and (N,R,μ_R,μ_N) are *FF*- modules, let

$$Hom_{\mu_R}(\mu_M,\mu_N)=\{\tilde{f}:\mu_M\to\mu_N\}.$$

We have:

Theorem 3.10 *Let* $\tilde{\phi}:\eta_R\to\mu_S$ *be a homomorphism of fuzzy ring,* (M,S,μ_S,μ_M) *and* (N,S,μ_S,μ_N) *be FF- modules, then*

$$Hom_{\mu_S}(\mu_M,\mu_N)<Hom_{\mu_R}(\mu_M,\mu_N).$$

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