Fuzzy quotient ring and isomorphic theorem

Yao Bingxue
Department of Mathematics
Liaocheng Teachers College
Shandong 252059 P.R.China

Abstract: In this paper we introduce the concept of fuzzy quotient ring of a ring and establish several isomorphic theorems.

Keywords: Fuzzy subring; fuzzy ideal; fuzzy quotient ring; homomorphism; isomorphism

1. Introduction

Since Liu introduced and studied fuzzy subring and fuzzy ideal many papers concerned with fuzzy ideal have been published. The purpose of this paper is to introduce the concept of fuzzy quotient ring and discuss the homomorphism and isomorphism of fuzzy quotient rings.

2. Preliminaries

Throughout this paper R and R' stand for rings and L denotes a complete distributive lattice.

Definition 2.1. A fuzzy subset of a nonempty set X is a mapping from X to L.

Definition 2.2. Let A be a fuzzy subset of R, if for all $x, y \in R$

(i) $A(x-y) \geqslant A(x) \wedge A(y)$

(ii) $A(xy) \geqslant A(x) \wedge A(y)$

then A is called a fuzzy subring of R. If condition (ii) is replaced by

(ii)′ $A(xy) \geqslant A(x) \vee A(y)$

then A is called a fuzzy ideal of R.

Proposition 2.3. Let A is a fuzzy ideal of R, then

$x+A = y+A \iff A(x-y) = A(0)$. In that case $A(x) = A(y)$.

Proposition 2.4. Let A and B be fuzzy subrings (fuzzy ideals) of R and R′, respectively, $f: R \to R'$ be a homomorphism, then

(i) $f(A)$ is a fuzzy subring (fuzzy ideal) of R′ and $f(A)(0)=A(0)$.

(ii) $f^{-1}(B)$ is a fuzzy subring (fuzzy ideal) of R which is constant on Kerf.

(iii) If f is an epimorphism, then $f(f^{-1}(B)) = B$.

(iv) If A is constant on Kerf, then $f^{-1}(f(A)) = A$.

Definition 2.5. Let A and B be fuzzy subrings of R and R′, respectively, $f: R \to R'$ be a homomorphism (isomorphism) from R onto R′. If $f(A) = B$, then f is called a homomorphism (isomorphism) from A onto B, we write

$$A \sim B \ (A \cong B).$$

3. Fuzzy quotient ring

Proposition 3.1. Let A and B be separately fuzzy subring and fuzzy ideal of R, then A/B is a fuzzy subring of R/B, where

$(A/B)(r+B) = \sup\{A(t) : t+B=r+B\}$

Proof. $\forall r_1, r_2 \in R$, $(A/B)((r_1+B)-(r_2+B)) = (A/B)(r_1-r_2+B)$

$= \sup\{A(x_1-x_2) : x_1-x_2+B=r_1-r_2+B\}$

$\geqslant \sup\{A(x_1) \wedge A(x_2) : x_1+B=r_1+B, x_2+B=r_2+B\}$

$= \sup\{A(x_1) : x_1+B=r_1+B\} \wedge \sup\{A(x_2) : x_2+B=r_2+B\}$

$= (A/B)(r_1+B) \wedge (A/B)(r_2+B)$

Similarly, $(A/B)((r_1+B)(r_2+B)) \geqslant (A/B)(r_1+B) \wedge (A/B)(r_2+B)$

Hence, A/B is a fuzzy subring of R/B.

Obviously, (i) $(A/B)(B) = A(0)$.

(ii) A/B is a fuzzy ideal of R/B if A is a fuzzy ideal of R.

Definition 3.2. Let A and B be separately fuzzy subring and fuzzy ideal of R, then A/B is called a fuzzy quotient ring of A concerned with B.

4. Isomorphic theorem

Definition 4.1. Let $R \overset{f}{\sim} R'$ and A be a fuzzy subring of R, then A^f is called the fuzzy kernel of A under f, where

$$A^f(r) = \begin{cases} A(0) & \text{if } r \in \mathrm{Ker} f \\ 0 & \text{otherwise} \end{cases}$$

Obviously, A^f is a fuzzy ideal of R and $A^f(0) = A(0)$.

Theorem 4.2. Let A and B be separately fuzzy subring and fuzzy ideal of R, then $A \sim A/B$.

Proof. It is clear that $f: R \rightarrow R/B$ is a homomorphism from R onto R/B, where $f(r) = r+B$.

$$f(A)(r+B) = \sup\{A(t) : f(t)=r+B\} = \sup\{A(t) : t+B=r+B\} = (A/B)(r+B)$$

That is, $f(A)=A/B$.

Hence, $A \sim A/B$.

Theorem 4.3. Let $R \overset{f}{\sim} R'$, A and B be fuzzy ideals of R and R', respectively, such that $A(0)>0$. If $A \sim B$, then $A/A^f \cong B$.

Proof. Let $g: R/A^f \rightarrow R'$ such that $g(r+A^f)=f(r)$,

then we can verify that g is an isomorphism from R/A^f onto R'.

$\forall y \in R'$, $g(A/A^f)(y) = \sup\{A/A^f(x+A^f) : g(x+A^f)=y\}$

$= \sup\{\sup\{A(t) : t+A^f=x+A^f\} : f(x)=y\}$

$= \sup\{A(t) : A^f(x-t)=A(0), f(x)=y\}$

$= \sup\{A(t) : x-t \in \mathrm{Ker} f, f(x)=y\} = \sup\{A(t) : f(t)=y\} = f(A)(y) = B(y)$

So, $g(A/A^f) = B$ and $A/A^f \cong B$.

Theorem 4.4. Let $R \overset{f}{\sim} R'$, A and B be separately fuzzy subring and fuzzy ideal of R. If B is constant on Kerf, then $A/B \cong f(A)/f(B)$.

Proof. Let $g: R/B \to R'/f(B)$ such that $g(r+B) = f(r)+f(B)$.

$\forall\ r_1, r_2 \in R$, we have

$$r_1+B = r_2+B \Longleftrightarrow B(r_1-r_2) = B(0)$$
$$\Longleftrightarrow f(B)(f(r_1)-f(r_2)) = B(r_1-r_2) = B(0) = f(B)(0)$$
$$\Longleftrightarrow f(r_1)+f(B) = f(r_2)+f(B)$$

Hence, g is a one-to-one mapping and further we can obtain $R/B \cong R'/f(B)$.

$\forall\ y+f(B) \in R'/f(B)$, $\exists x \in R$, such that $f(x)=y$,

$$\begin{aligned} g(A/B)(y+f(B)) &= \sup\{(A/B)(r+B):\ g(r+B)=y+f(B)\} \\ &= \sup\{(A/B)(r+B):\ f(r)+f(B)=f(x)+f(B)\} \\ &= \sup\{(A/B)(r+B):\ B(x-r)=B(0)\} \\ &= \sup\{\sup\{A(t):\ t+B=r+B\} : B(x-r)=B(0)\} \\ &= \sup\{A(t):\ t+B=x+B\} \end{aligned}$$

$$\begin{aligned} (f(A)/f(B))(y+f(B)) &= \sup\{f(A)(u):\ u+f(B)=y+f(B)\} \\ &= \sup\{\sup\{A(v):\ f(v)=u\} : u+f(B)=y+f(B)\} \\ &= \sup\{A(v):\ f(v)+f(B)=f(x)+f(B)\} \\ &= \sup\{A(v):\ v+B=x+B\} \end{aligned}$$

That is, $g(A/B) = f(A)/f(B)$

Hence, $A/B \cong f(A)/f(B)$.

Corollary 4.5. Let $R \overset{f}{\sim} R'$, A and B be separately fuzzy subring and fuzzy ideal of R', then $f^{-1}(A)/f^{-1}(B) \cong A/B$.

Lemma 4.6. Let A and B be fuzzy ideals of R such that $A \geqslant B$ and $A(0)=B(0)$, then for all $x, y \in R$

$$x+B+A/B = y+B+A/B \Longleftrightarrow x+A = y+A.$$

Proof. $x+B+A/B = y+B+A/B \Longrightarrow (A/B)(x-y+B) = (A/B)(B) = A(0)$

$$\Longrightarrow \sup\{A(t):\ t+B=x-y+B\} = A(0)$$

For all $t \in \{t: t+B=x-y+B\}$ we have $B(x-y-t) = B(0)$ and

$A(x-y) \geqslant A(x-y-t) \wedge A(t) \geqslant B(x-y-t) \wedge A(t) = A(t)$.

So, $A(x-y) \geqslant \sup\{A(t):\ t+B=x-y+B\} = A(0)$

That is, $A(x-y) = A(0)$, $x+A = y+A$.

Inversely, $x+A = y+A \Longrightarrow A(x-y) = A(0)$

$\Longrightarrow A/B(x-y+B) = \sup\{A(t):\ t+B=x-y+B\} \geqslant A(x-y) = A(0) = A/B(B)$

$\Longrightarrow x+B+A/B = y+B+A/B$.

Hence, $x+B+A/B = y+B+A/B \Longleftrightarrow x+A = y+A$.

Theorem 4.7. Let A and B be fuzzy ideals of R such that $A \geqslant B$ and $A(0)=B(0)$. If C is a fuzzy subring of R, then $(C/B)/(A/B) \cong C/A$.

Proof. Let $f:(R/B)/(A/B) \to R/A$ such that $f(r+B+A/B)=r+A$

From Lemma 4.6 we can prove that f is an isomorphism from $(R/B)/(A/B)$ onto R/A. $\forall r+B+A/B \in (R/B)/(A/B)$

$$
\begin{aligned}
(C/B)/(A/B)(r+B+A/B) &= \sup\{(C/B)(x+B):\ x+B+A/B=r+B+A/B\} \\
&= \sup\{\sup\{C(t):\ t+B=x+B\} : x+B+A/B=r+B+A/B\} \\
&= \sup\{C(t):\ t+B+A/B=r+B+A/B\} \\
&= \sup\{C(t):\ t+A=r+A\} = (C/A)(r+A) \\
&= (C/A)(f(r+B+A/B)) = f^{-1}(C/A)(r+B+A/B)
\end{aligned}
$$

Hence $(C/B)/(A/B) = f^{-1}(C/A)$.

From Proposition 2.5 we obtain $f((C/B)/(A/B)) = C/A$

That is, $(C/B)/(A/B) \cong C/A$.

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