# Intersection Decomposition and Monotonic Intersection Decomposition of Fuzzy Sets

*Liu Huawen and Li Jiuping*

Department of Mathematics and Physics, Shandong University of Technology,

Jinan 250061, P.R.China

Abstract: The new decompositions and the monotonic decompositions of fuzzy sets are established by means of t-conorm S. Further, the corresponding decomposition and monotonic decomposition of 2-type fuzzy sets are given.

Keywords: T-conorm, simple fuzzy set, intersection decomposition, monotonic intersection decomposition.

## 1. Decomposition theorem.

Let X be a set, F(X) be a set of fuzzy subsets of X and P (X) be a set of subsets of X. For any $A \in F(X)$, $A(x)$ denotes a membership function of A.

A t-norm T is defined as a function $T: [0,1] \times [0,1] \to [0,1]$ satisfying the following properties (See [1] for details):

(i) $T(x,1)=x$,

(ii) $T(x,y) \leq T(z,u)$ if $x \leq z$ and $y \leq u$,

(iii) $T(x,y)=T(y,x)$,

(iv) $T(x,T(y,z))=T(T(x,y),z)$,

for all x,y,z and u in [0,1].

A t-conorm S associated with t-norm T is defined by

$$S(x,y)=1-T(1-x,1-y) \quad \text{for all } x,y \in [0,1].$$

Some useful properties of t-conorm are:

$S(x,y) \geq x \vee y$; $S(x,y) \leq S(z,u)$ if $x \leq z$ and $y \leq u$.

We write $A_\lambda = \{x \mid A(x) \geq \lambda, x \in X\}$, $A_{\underline{\lambda}} = \{x \mid A(x) > \lambda, x \in X\}$

Theorem 1.1 (Decomposition theorem I) Let $A \in F(X)$, then

$$A(x) = \bigwedge_{\lambda \in [0,1]} S(\lambda, A_\lambda(x)) \quad (\forall x \in X) \qquad (1.1)$$

Proof. $\bigwedge_{\lambda \in [0,1]} S(\lambda, A_\lambda(x)) = \bigwedge_{\lambda \in [0,1]} \{\lambda \mid A(x) < \lambda\} = A(x)$

Theorem 1.2 (Decomposition theorem II) Let $A \in F(X)$, then

$$A(x) = \bigwedge_{\lambda \in [0,1]} S(\lambda, A_{\underline{\lambda}}(x)) \quad (\forall x \in X)$$

Proof. Same to theorem 1.1.

Using theorem 1.1 and theorem 1.2, we get

Theorem 1.3 (Decomposition theorem Ⅲ) Let $A \in F(X)$, $H:[0.1] \to P(X)$, $\lambda \mapsto H(\lambda)$ satisfy $A_{\underline{\lambda}} \subseteq H(\lambda) \subseteq A_\lambda$ $(\lambda \in [0,1])$, then

1) $A(x) = \bigwedge_{\lambda \in [0,1]} S(\lambda, H(\lambda)(x))$ (1.3)

2) $\lambda_1 < \lambda_2 \Rightarrow H(\lambda_1) \supseteq H(\lambda_2)$ (1.4)

3) $A_\lambda = \bigcap_{\alpha < \lambda} H(\alpha)$ $(\lambda \neq 0)$, $A_{\underline{\lambda}} = \bigcup_{\alpha > \lambda} H(\alpha)$ $(\lambda \neq 1)$ (1.5)

2.Montonic decomposition theorem

Definition 2.1 $\{E_1, E_2, \cdots, E_n\}$ is called a divide of X, if $X = \bigcup_{i=1}^{n} E_i$, $E_i \cap E_j = \Phi$ $(i \neq j)$. A is called a simple fuzzy set, if $A(x) = \lambda_i$ $(\lambda_i \in [0,1])$ for all $x \in E_i$.

Definition 2.2 Let A, $A^{(n)} \in F(X)$, A is called the limit of the sequence $A^{(n)}$ of fuzzy sets, if for each $x \in X$, there exists $\lim_n A^{(n)}(x) = A(x)$. We write $\lim_n A^{(n)} = A$.

Definition 2.3 Let $A^{(n)} \in F(X)$, $n=1,2,\cdots$. $\{A^{(n)}\}$ is called a monotonic decreasing sequence of fuzzy sets, if $A^{(n)} \supseteq A^{(n+1)}$, $n=1,2,\cdots$. Where $A^{(n)} \supseteq A^{(n+1)}$ implies $A^{(n)}(x) > A^{(n+1)}(x)$ for all $x \in X$.

Lemma Let $\{A^{(n)}\}$ be a monotonic decreasing sequence of fuzzy sets. Then $\lim_n A^{(n)}(x)$ exists for any $x \in X$, and $\lim_n A^{(n)} = \bigcap_n A^{(n)}$.

Proof is omitted.

Theorem 2.1 (Monotonic decomposition theorem) Let $A \in F(X)$, then there exists a monotonic decreasing sequence $\{A^{(n)}\}$ of simple fuzzy sets, such that $A = \bigcap_n A^{(n)}$.

Proof. Make $A^{(n)}(x) = \frac{k}{2^n}$

Where $x \in E_{n,k} = \{x \mid \frac{k-1}{2^n} < A(x) \leq \frac{k}{2^n}, x \in X\}$, $k=1,2,\cdots,2^n$.

$x \in E_{n,0} = \{x \mid A(x) = 0, x \in X\}$, $n=1,2,\cdots$

Then $A^{(n)} \in F(X)$ and $A^{(n)}$ is a simple fuzzy set.

We now prove the following two points.

1) $\{A^{(n)}\}$ is a monotonic decreasing sequence. i.e., for any $x \in X$, there exists $A^{(n)}(x) > A^{(n+1)}(x)$.

$$E_{n,k}=\{x \mid \frac{2(k-1)}{2^{n+1}}<A(x)\leq\frac{2k-1}{2^{n+1}},\ x\in X\}\cup\{x \mid \frac{2k-1}{2^{n+1}}<A(x)\leq\frac{2k}{2^{n+1}}, x\in X\}$$

$$=E_{n+1,2k-1}\cup E_{n+1,2k}.$$

For any $x\in E_{n,k}$,

1° if $x\in E_{n+1,2k-1}$, then $A^{(n)}(x)>A^{(n+1)}(x)$;

2° if $x\in E_{n+1,2k}$, then $A^{(n)}(x)=A^{(n+1)}(x)$.

i.e., for any $x\in E_{n,k}$, $A^{(n)}(x)\geq A^{(n+1)}(x)$. Therefore, it is the same on X.

2) $A=\lim_n A^{(n)}$. i.e. for any $x\in X$, there exists $\lim_n A^{(n)}(x)=A(x)$.

For any $x\in X$,

1° if $x\in E_{n,0}=\{x \mid A(x)=0\}$, then $A^{(n)}(x)=0$, i.e. $\lim_n A^{(n)}(x)=A(X)$;

2° if $x\notin E_{n,0}$, then $0<A(x)\leq 1$. For any n, there exists $k_0(n)$, such that

$$\frac{k_0(n)-1}{2^n}<A(x)\leq\frac{k_0(n)}{2^n}.$$

Thus $|A^{(n)}(x)-A(x)|\leq\frac{1}{2^n}$. i.e. $\lim_n A^{(n)}(x)=A(x)$.

Therefore $A=\lim_n A^{(n)}$. From lemma, we obtain $A=\bigcap_n A^{(n)}$.

*In the following, we give several methods constructing* $A^{(n)}$.

Theorem 2.2 Let $A\in F(X)$. We make a sequence $A^{(n)}$ of simple fuzzy sets:

$$A^{(n)}(x)=\bigwedge_{k=0}^{2^n} S(\frac{k}{2^n},A_{\underline{\frac{k}{2^n}}}(x))\quad(\forall x\in X)\ n=1,2,\cdots.$$

Then

1) $A^{(n)}\supseteq A^{(n+1)}$, $n=1,2,\cdots$.

2) $A=\bigcap_n A^{(n)}$.

Theorem 2.3 Let $A\in F(X)$. We make a sequence $A^{(n)}$ of simple fuzzy sets:

$$A^{(n)}(x)=\bigwedge_{k=0}^{2^n} S(\frac{k}{2^n},A_{\underset{\cdot}{\frac{k}{2^n}}}(x))\quad(\forall x\in X)\ n=1,2,\cdots.$$

The proofs of theorem 2.2 and 2.3 are similar to theorem 2.1. From theorem 2.2 and theorem 2.3, we get

Theorem 2.4 Let $A\in F(X)$ and $H:[0,1]\to P(X)$, $\lambda\mapsto H(\lambda)$ satisfy

$$A_{\underset{\cdot}{\lambda}}\subseteq H(\lambda)\subseteq A_\lambda\quad(\lambda\in[0,1])$$

We make a sequence $A^{(n)}$ of simple fuzzy sets:

$$A^{(n)}(x)=\bigwedge_{k=0}^{2^n} S(\frac{k}{2^n},H(\frac{k}{2^n})(x)),\ (\forall x\in X),n=1,2,\cdots.$$

Then

1) $A^{(n)}\supseteq A^{(n+1)}$, $n=1,2,\cdots$.

2) $A=\bigcap_n A^{(n)}$.

3. Decomposition and monotonic decomposition of 2-type fuzzy sets

In this section, the proofs of all theorems are similar to section 1 and section 2 respectively.

Let $F_{[0,1]}(X)=\{A \mid A:X\rightarrow F([0,1])\}$ be a set of 2-type fuzzy sets on X. We write $A_{(t,\lambda)}=\{x \mid A(x)(t)\geq\lambda\}$, $\underline{A}_{(t,\lambda)}(x)=\{x \mid A(x)(t)>\lambda\}$, $\forall t,\lambda\in[0,1]$.

Theorem 3.1 (Decomposition theorem) Let $A\in F_{[0,1]}(X)$. Then

1) $A(x)(t)=\bigwedge_{\lambda\in[0,1]}S(\lambda,A_{(t,\lambda)}(x))$ $\forall x\in X,t\in[0,1]$.

2) $A(x)(t)=\bigwedge_{\lambda\in[0,1]}S(\lambda,\underline{A}_{(t,\lambda)}(x))$ $\forall x\in X,t\in[0,1]$.

3) For any $t\in[0,1]$, let $H:[0,1]\rightarrow P(X)$, $\lambda\mapsto H(t,\lambda)$ satisfy

$$\underline{A}_\lambda\subseteq H(t,\lambda)\subseteq A_\lambda\quad(\lambda\in[0,1])$$

Then

i) $A(x)(t)=\bigwedge_{\lambda\in[0,1]}S(\lambda,H(t,\lambda)(x))$

ii) $\forall t\in[0,1]$, $\lambda_1<\lambda_2\Rightarrow H(t,\lambda_1)\supseteq H(t,\lambda_2)$

iii) $\forall t\in[0,1]$, $A_{(t,\lambda)}=\bigcap_{\alpha<\lambda} H(t,\alpha)\quad(\lambda\neq0)$

$$\underline{A}_{(t,\lambda)}=\bigcup_{\alpha>\lambda} H(t,\alpha)\quad(\lambda\neq1)$$

Definition 3.1 Let $A^{(n)},A\in F_{[0,1]}(X)$, $n=1,2,\cdots$. $\lim_n A^{(n)}=A$ implies $\lim_n A^{(n)}(x)=A(x)$, for any $x\in X$.

Definition 3.2 Let $A,B\in F_{[0,1]}(X)$. $A\supseteq B$ implies $A(x)\supseteq B(x)$ for any $x\in X$.

Theorem 3.2 (Monotonic decomposition) Let $A,B\in F_{[0,1]}(X)$, then exists $A^{(n)}\in F_{[0,1]}(X)$, $n=1,2,\cdots$, such that 1) $A^{(n)}\supseteq A^{(n+1)}$, $n=1,2,\cdots$.

2) $A=\bigcap_n A^{(n)}$.

Definition 3.3 Let $\{E_i,\ i=1,2,\cdots,n\}$ve a divide of X. We call 2-type fuzzy set A is a simple 2-type fuzzy set, if $A(x)=\lambda_i$ for all $x\in E_i$, $(\lambda_i\in F([0,1]))$ $i=1,2,\cdots,n$.

The following theorems give some methods constructing $A^{(n)}$.

**Theorem 3.3** Let $A\in F_{[0,1]}(X)$. We make a sequence $A^{(n)}$ of simple 2-type fuzzy sets:

$$A^{(n)}(x)(t)=\bigwedge_{k=0}^{2^n} S(\frac{k}{2^n},\ A_{(t,\frac{k}{2^n})}(x)),\forall x\in X,t\in[0,1].$$

or

$$A^{(n)}(x)(t)=\bigwedge_{k=0}^{2^n} S(\frac{k}{2^n},\ \underset{\cdot}{A}_{(t,\frac{k}{2^n})}(x)),\forall x\in X,t\in[0,1].$$

Then 1) $A^{(n)}\supseteq A^{(n+1)}$, $n=1,2,\cdots$.

2) $A=\bigcap_n A^{(n)}$.

**Theorem 3.4** Let $A\in F_{[0,1]}(X)$ and for any $t\in[0,1]$,

$H:[0,1]\to P(X),\lambda \mapsto H(t,\lambda)$

satisfy $A_{\underset{\cdot}{\lambda}}\subseteq H(t,\lambda)\subseteq A_\lambda \quad (\lambda\in[0,1])$.

We make a sequence $A^{(n)}$ of simple 2-type fuzzy sets:

$$A^{(n)}(x)(t)=\bigwedge_{k=0}^{2^n} S(\frac{k}{2^n},H(t,\frac{k}{2^n})(x)),\forall x\in X,t\in[0,1].$$

Then 1) $A^{(n)}\supseteq A^{(n+1)}$, $n=1,2,\cdots$.

2) $A=\bigcap_n A^{(n)}$.

Similar to above methods, we can establish the intersection decomposition and monotonic intersection decomposition of high-type fuzzy sets

References


[1] B.Schweizer ad A.Sklar.Probabilistic Mettric Spaces.(North-Holland,Amsterdan,1983).

[2] D. Dubois and H. Prade. Fuzzy Sets and Systems: Theory and Application. (Academic Press,New York,1980).

[3] Gu Wenxiang and Chen Degang.Intersection decomposition of fuzzy sets,Fuzzy systems and Mathematics. (Changsha,China),8(1994),414-415.

[4] Liu Huawen.On Basic Theory of Fuzzy Sets. Journal of Shandong Normal University,11 (1996),70-72.

[5] Lin Zongzhen. Monotonic decomposition theorem of fuzzy sets. Journal of Jinan University, (Guangzhou,China),16(1)(1995),26-29.

[6] L.A.Zadeh.Fuzzy Sets.Information and Control,8(1965),338-358.

[7] Luo Chengzhong. Introduction to Fuzzy Sets.(Beijing Normal University Press,China, 1989).