NEW FAMILY OF GIRARD-MONOIDS ON INTERVALS VIA ANNIHILATION

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Abstract

In this paper a certain transformation of triangular-norms (called n-annihilation) is studied. This problem is strongly related to the contrapositive symmetry of residuated implications. We characterize those continuous triangular-norms where the annihilated binary operation is triangular-norm. Some surprising properties of nilpotent triangular-norms are presented e.g. the nilpotent minimum is described as limit of nilpotent triangular-norms. As a consequence, a new family of triangular-norms (called nilpotent ordinal sums) owing several attractive properties is discovered. The new family contains the nilpotent triangular-norms and the nilpotent minimum as limit cases and can be admitted into investigations in the theory of Girard-monoids as well.

1 Introduction

A triangular norm (t-norm for short) is a function T from $[0,1]^2$ to $[0,1]$ being commutative, associative, nondecreasing in each place and $T(1,x) = x$ holds for all $x \in [0,1]$. Throughout this paper n denotes a strong negation (i.e., an involutive order reversing bijection of the closed unit interval).

Let T be a t-norm. Define the binary operation $T_{(n)}$ as follows:

$T_{(n)} \; : \; [0,1] \times [0,1] \to [0,1]$;

$$T_{(n)}(x,y) = \begin{cases} T(x,y) & \text{if } x > n(y) \\ 0 & \text{if } x \leq n(y) \end{cases} . \tag{1}$$

In this paper we characterize those continuous t-norms T where $T_{(n)}$ is a t-norm too. Fodor has given the following examples for this problem in [1]:

1.) If $T(x,y) = \min(x,y)$ then $T_{(n)}$ is a t-norm.

2.) If $T(x,y) = x \cdot y$ then $T_{(n)}$ is not associative, hence not a t-norm.

The origin of the problem can be found in [1] in the following context.

1.1 Contrapositive symmetrization of residuated implications

The residuated implication function $I_T : [0,1] \times [0,1] \to [0,1]$ generated by the t-norm T is defined via

$$I_T(x,y) := \sup\{z \in [0,1] : T(x,z) \leq y\}.$$

We say that a binary operation I has the *contrapositive symmetry* property with respect to n if and only if

$$I(x,y) = I(n(y), n(x)) \tag{2}$$

holds for all $x, y \in [0,1]$. Generally, a residuated implication (generated by a t-norm) doesn't fulfil this property. Then we can symmetrize it as follows:

Suppose that T is a left-continuous t-norm. Define a new operation associated with I_T:

$$x \rightarrow_T y = \max\{I_T(x,y), I_T(n(y), n(x))\}.$$

Define also a binary operation $*_T$ by

$$x *_T y = \min\{T(x,y), n[I_T(y, n(x))]\}.$$

$*_T$ is a fuzzy conjunction having nice properties but generally, it is not a t-norm. $\rightarrow_T$ always has the contrapositive symmetry property w.r.t. n and this is the fuzzy implication generated by $*_T$ via residuation (see Theorem 2. in [1]). So, if $*_T$ is a t-norm T then $\rightarrow_T = I_T$ and I_T admits the contrapositive symmetry property. More precisely I_T has the contrapositive symmetry property w.r.t. $I_T(x,0)$. Now the question arises naturally: Under which conditions can we state that $*_T$ is a t-norm ? For a left-continuous t-norm T the residuated implication defines an order on the closed unit interval (i.e., $x \le y$ if and only if $I_T(x,y) = 1$) so we can see that $x *_T y = 0$ if $y \le n(x)$. The definition of $*_T$ yields as well that $x *_T y \le T(x,y)$ if $y > n(x)$. Obviously,

1.) $*_T$ coincides with $T_{(n)}$ defined in (1)

if and only if

2.) $T(x,y) \le n[I_T(y, n(x))]$ holds for $y > n(x)$.

Therefore, solving equality (1) gives us a chance to find new t-norms, for which the residuated implication generated by them admits (2).

1.2 Girard-monoids

Another important application is related to the theory of integral commutative residuated lattice ordered monoids. Let $(L, \vee, \wedge, 1, 0)$ be a lattice with greatest element 1 and least element 0 and $(L, \sqcap)$ be a commutative semigroup with unit 1 and zero 0. ($\sqcap$ is used as the model of "and".) Consider the structure $\mathcal{L} = (L, \vee, \wedge, 1, 0, \sqcap)$. Suppose that the distributivity law holds, i.e., for all $a, b, c \in L$

$$a \sqcap (b \vee c) = (a \sqcap b) \vee (a \sqcap c).$$

Now define the implication via residuation. In other words, suppose that all residuals exist:

$$x \rightarrow_\sqcap y := \sup\{z \in L \mid x \sqcap z \le y\}$$

Then introduce residual complements via

$$x' := x \rightarrow_\sqcap 0.$$

Clearly, $0' = 1$ and $1' = 0$. Finally, define the dual semigroup operation with the help of the De Morgan Law as follows:

$$x \sqcup y := (x' \sqcap y')'.$$

Then the involutive property of $'$ plays an important role. Namely, consider the following:

$$a \sqcup 0 = (a' \sqcap 0')' = (a' \sqcap 1)' = (a')'.$$

Therefore, the involutive property of $'$ is equivalent to the property $a \sqcup 0 = a$, which is a crucial point of the construction. If $(a')' = a$ holds then $\mathcal{L}$ is called a *Girard-monoid*.

Turning back to the most important mathematical application, where $\mathcal{L} = [0,1]$ and the logical connective "and" is modelled by a t-norm it yields the following problem: Characterize those left-continuous t-norms T where $I_T(x,0)$ is involutive.

Among continuous t-norms the answer is known (see e.g. Section 1.8.2 in [3]). These are the t-norms which are φ-transformations of the Lukasiewicz t-norm which is defined by the following formula:

$$W(x,y) = \max\{0, x+y-1\}.$$

In other words the nilpotent t-norms.
Another example is the nilpotent minimum (see [1]), which is defined by the following formula:

$$\min_0(x,y) := \begin{cases} \min(x,y) & \text{if } y > n(x) \\ 0 & \text{otherwise} \end{cases}.$$

It was proved in [8] that any strong negation n can be represented in the following form

$$n(x) = \varphi^{-1}(1-\varphi(x)), \tag{3}$$

where φ is an automorphism of the closed unit interval. Using this representation of n, the following formula is obtained for nilpotent minimum with respect to n:

$$\min_{0,\varphi}(x,y) = \begin{cases} 0 & \text{if } \varphi(x)+\varphi(y) \le 1 \\ \min(x,y) & \text{if } \varphi(x)+\varphi(y) > 1 \end{cases}.$$

This paper presents a wide class of new t-norms with the desired property via solving the problem mentioned at the beginning of this section.

1.3 Left-continuous but not continuous t-norms

In [3] the authors' conjecture is that the nilpotent minimum (up to an automorphism) is the only t-norm which is left-continuous but not continuous. It is not so, since e.g. an ordinal sum (see [6]) defined by one nilpotent minimum summand is as well left-continuous and not continuous and not isomorphic to the nilpotent minimum. However, examples of this type are trivial counterexamples since all of them are built with the help of the nilpotent minimum itself. Clearly, any discontinuous solution $T_{(n)}$ of the equation (1) which is different from the nilpotent minimum, gives a non-trivial example for left-continuous but not continuous t-norms.

1.4 The annihilation problem

Several recent papers deal with the problem of finding methods which produce new t-norms out of known t-norms (e.g. [5]). Therefore, solving (1) has its own interest. We call the method, which produces $T_{(n)}$ from T *n-annihilation.*

The paper is organized as follows: In the next section a brief preliminary is given. Then in Section 3 we investigate the relation between the annihilation problem and the other problems which were mentioned in the introduction. In Section 4 we present the main theorems of this paper. Characterization of those continuous t-norms where the annihilated operation is a t-norm is given. In Section 5 we present some facts about continuous Archimedean t-norms. The introduction and investigation of the nilpotent ordinal sums is presented in section 6. Finally, the conlusion of the paper is given in Section 7.

2 Preliminaries

A t-norm is said to be *continuous* if it is continuous as a two-place function. A continuous t-norm T is called *Archimedean* if $T(x,x) < x$ is true for all $x \in (0,1)$. A t-norm T has *0-divisors* if $T(x,y) = 0$ for some $x, y \in (0,1)$. A continuous Archimedean t-norm with 0-divisors is called *nilpotent.* If φ is an *automorphism* that is, an increasing bijection of the closed unit interval, then the following formula defines the so called *φ-transform* of T (which is as well a t-norm):

$$T_\varphi(x,y) = \varphi^{-1}(T(\varphi(x),\varphi(y))), \qquad x,y \in [0,1].$$

Suppose that $\{[a_i,b_i]\}_{i\in K}$ is a countable family of non-overlapping, closed, proper subintervals of [0,1], denoted by $\mathcal{I}$. With each $[a_i,b_i] \in \mathcal{I}$ associate a continuous Archimedean t-norm T_i. Let T be a function defined on $[0,1]^2$ by

$$T(x,y) = \begin{cases} a_i + (b_i - a_i)T_i\left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}\right) & \text{if } (x,y) \in [a_i,b_i]^2 \\ \min(x,y) & \text{otherwise} \end{cases}. \qquad (4)$$

In this case T is denoted by $\Pi\{([a_i,b_i],T_i)\}_{i\in K}$ and called the *ordinal sum* of $\{([a_i,b_i],T_i)\}_{i\in K}$ and each T_i is called a *summand.* Ling's theorem [6] says that a t-norm is continuous if and only if it is (a possibly empty) ordinal sum of continuous Archimedean t-norms.

3 The relation between the annihilation problem and the other problems in Section 1

In this section we point to the fact that the annihilation problem, which was described at Subsection 1.4 is equivalent to the problem which was described at Subsection 1.1. That is, finding all the left-continuous t-norms which can be n-annihilated is equivalent of the finding all those left-continuous t-norms where the residuated implication generated by them admits the contrapositive symmetry property with respect to a strong negation n. Moreover, in the more general structure (integral commutative residuated lattice ordered monoids) the (suitaby extended) annihilation problem leads exactly to the problem which was described in Subsection 1.2. That is, the involutive property of $'$ is guaranteed if and only if $\sqcap$ is a solution of the (extended) annihilation problem.

The following proposition and its corollary are consequences of some results in [4].

Proposition 1 *Suppose that $\mathcal{L}$ is a Girard-monoid. Define a new implication as follows (this implication is often referred to as S-implication in the literature):*

$$x \to_\sqcup y = x' \sqcup y.$$

Then for all $x,y \in L$ it holds true that $x \to_\sqcup y = x \to_\sqcap y$.

Corollary 1 *Let $\mathcal{L}$ be a Girard-monoid. Then $\to_\sqcap$ admits the contrapositive symmetry property with respect to $'$. That is, $x \to_\sqcap y = y' \to_\sqcap x'$ holds for all $x,y \in L$.*
The previous corollary says that the contrapositive symmetry of $\to_\sqcap$ restricted to 0 (i.e., $x \to_\sqcap 0 = 0' \to_\sqcap x' = 1 \to_\sqcap x' = x'$) yields the contrapositive symmetry of $\to_\sqcap$ with all y's (i.e., $x \to_\sqcap y = y' \to_\sqcap x'$).

Corollary 2 *Let T be a left-continous t-norm. Then I_T admits the contrapositive symmetry property with respect to a strong negation n if and only if $I_T(x,0)$ is involutive. In this case $n(x)$ is equal to $I_T(x,0)$.*

4 The characterization

Denote by t the unique solution of $n(x) = x$, where n is a strong negation and $x \in [0,1]$. Now we characterize those continuous Archimedean t-norms, which can be n-annihilated (i.e., the n-annihilation remains a t-norm).

Theorem 1 *Let T be a continuous Archimedean t-norm. Then T can be n-annihilated if and only if T admits the Law of contradiction w.r.t. n (i.e., $T(x,n(x)) = 0$ holds for all $x \in [0,1]$).*

Definition 1 A t-norm T is said to be a *trivial annihilation* (with respect to the strong negation n) if $n(x) = I_T(x,0)$ for $x \in [0,1]$. One can see easily that if a continuous t-norm T is trivial annihilation (with respect to the strong negation n) then $T(x, n(x)) = 0$ for all $x \in [0,1]$ and hence the binary operation $T_{(n)}$ defined in (1) is equal to T. The opposite implication is generally not true since $T(x, n(x)) = 0$ implies only $n(x) \leq I_T(x,0)$.

Definition 2 Let T_1 and T_2 be t-norms. T_1 is said to be *similar* to T_2 with respect to the annihilation along the strong negation n ($T_1 \leftrightarrow_n T_2$) if

$$T_1|_{\{(x,y)\in[0,1]^2 \mid x>n(y)\}} = T_2|_{\{(x,y)\in[0,1]^2 \mid x>n(y)\}}.$$

Obviously, $T_1 \leftrightarrow_n T_2$ if and only if $T_{1(n)} = T_{2(n)}$ and it is clearly an equivalence relation.

Definition 3 Let T be a continuous non-Archimedean triangular norm, let $([a,b], T_1)$ be a summand of T. We say that this summand is *in the center* (w.r.t. the strong negation n) if $a = n(b)$. (The name is based on the observation that if we evaluate $n(x) = 1 - x$ then this condition yields that $[a,b] \times [a,b]$ is indeed in the center of the unit square.)

Now, for each strong negation n and for each summand $([a,b], T_1)$ which is in the center w.r.t. n, we define a stong negation n_a^b as follows ($x \in [0,1]$):

$$n_a^b(x) = \frac{n\left(x \cdot (b-a) + a\right) - a}{b-a}. \tag{5}$$

Straightforward calculation shows that n_a^b is indeed a strong negation. Notice, that this negation is "a part" of the negation n in the sense that we zoomed $n|_{[a,b]}$ to $[0,1] \times [0,1]$. In other words, this is the "negation" which goes inside the summand. As we will see this negation plays an important role in the characterization. Now we present the main theorem of this paper. This theorem together with Theorem 1 gives the characterization of the continuous t-norms which can be n-annihilated.

Theorem 2 *Let T be a continuous non-Archimedean t-norm. Then $T_{(n)}$ is a t-norm if and only if*
1.) $T \leftrightarrow_n \min$ *or*
2.) T is similar to a t-norm which is defined by one trivial annihilation summand in the center. More formally, $T \leftrightarrow_n \Pi\{([a,b], T_1)\}$ where $a < b$, $a > 0$, $a = n(b)$ and T_1 is trivial annihilation w.r.t. to the negation n_a^b.

5 On continuous Archimedean t-norms

Althought easy calculation reveals, it is not well-known that not only the Lukasiewicz t-norm is trivial annihilation w.r.t. $1 - x$. Now, we characterize those continuous Archimedean t-norms which are trivial annihilation w.r.t. n. In other words, those t-norms for which $I_T(x,0) = n(x)$ holds. That is, those t-norms which has positive values exactly in the upper right "triangle" of the unit square limited by the graph of n.

Definition 4 We call an automorphism *symmetric* if its graph is centrally symmetric to the point $(\frac{1}{2}, \frac{1}{2})$.

Let T_1 and T_2 be two nilpotent t-norms. That is, $T_1 = W_{\varphi_1}$, $T_2 = W_{\varphi_2}$ for some automorphisms φ_1 and φ_2 of the closed unit interval, respectively (see [7]).

Proposition 2 *The automorphisms are unique.*
It is well-known from Corollary 1,2. in [1] that $I_{T_1}(x,0) = \varphi_1^{-1}(1 - \varphi_1(x))$ and $I_{T_2}(x,0) = \varphi_2^{-1}(1 - \varphi_2(x))$. Denote these two negations by n_1 and n_2, respectively.

Proposition 3 *$n_1 = n_2$ if and only if $\psi = \varphi_1 \circ \varphi_2^{-1}$ is a symmetric automorphism.*
Proposition 3 says that having a t-norm T with the negation $n(x) = I_T(x,0)$ the negation $(n_\varphi(x) = I_{T_\varphi}(x,0))$ of T_φ remains $n(x)$ if and only if φ is a symmetric automorphism.

As another surprising fact, we present now that the nilpotent minimun can be described as the limit of continuous Archimedean t-norms, where each continuous Archimedean t-norm has the trivial annihilation property w.r.t. $1-x$. This means that the graph of a continuous Archimedean t-norm can be very similar to the graph of the nilpotent minimum.

Theorem 3 *There exists a sequence of continuous Archimedean t-norms T_k $(k = 1, 2...)$ such that $\lim_{k\to\infty} T_k(x,y) = \min_0(x,y)$. Moreover, for all k, T_k is trivial annihilation w.r.t. $1-x$.*

Remark 1 In order to give another impression about the wide variety of continuous Archimedean t-norms we recall that any continuous t-norm (that is, an ordinal sum with continuous Archimedean summands) is a uniform limit of continuous Archimedean t-norms (both nilpotent or strict ones). For detailes see [2].

6 Nilpotent ordinal sums

Let $n(x)(= \varphi^{-1}(1-\varphi(x)))$ be a strong negation, t its unique fixed point, $a \in [0,t]$ and T_1 is trivial annihilation t-norm w.r.t. $n_a^{n(a)}$.

Define

$$T_{n,a,T_1}(x,y) = \begin{cases} 0 & \text{if } x \le n(y) \\ a + (n(a)-a) \cdot T_1\left(\frac{x-a}{n(a)-a}, \frac{y-a}{n(a)-a}\right) & \text{if } a \le x, y \le n(a) \text{ and } x > n(y) \\ \min(x,y) & \text{otherwise} \end{cases} . \quad (6)$$

If $a \in (0,t)$ then a t-norm of this type is called a *nilpotent ordinal sum* with respect to the strong negation n. Without any restriction for the parameter a it is called a nilpotent ordinal sum in the extended sense.

Consider the following family:

$$\mathcal{J} = \{T \mid T = T_{n,a,T_1} \ , \ a \in (0,t)\} .$$

Theorem 2 yields that any element of $\mathcal{J}$ is a t-norm. A tipical nilpotent ordinal sum has the following structure (see Figure 1.).

Figure 1.

Theorem 4 *For any $T = T_{n,a,T_1} \in \mathcal{J}$ the residuated implication generated by T admits the contrapositive symmetry property with respect to n. The residuated implication generated by T has the following form:*

$$I_T(x,y) = \begin{cases} 1 & \textit{if } x \le y \\ \max\left(a + (n(a)-a) \cdot I_{T_1}\left(\frac{x-a}{n(a)-a}, \frac{y-a}{n(a)-a}\right), n(x)\right) & \textit{if } a \le x \le n(a) \textit{ and } x > y \\ \max(y, n(x)) & \textit{otherwise} \end{cases} . \quad (7)$$

Theorem 5 *Any two members of the family $\mathcal{J}$ are isomorphic to each other.*

This family of t-norms and the corresponding implications have not been known in the literature yet. Concerning the contrapositive symmetry of fuzzy implications, from [1] we know that a residuated implication which was generated by a left-continuous t-norm has the contrapositive symmetry property w.r.t. n if and only if the R-implication (i.e., I_T) and the S-implication (i.e., $S(n(x), y) = n(T(x, n(y)))$) coincide. So far, two examples were known for that.

1.) The trivial annihilation continuous Archimedean t-norms and the
2.) nilpotent minimum.

According to Theorem 4 there is a third family:
3.) the family of nilpotent ordinal sums.

This family can be described as a bridge between the first two ones. Indeed, one can check easily that

$$\lim_{a\to t} T_{n,a,T_1} = \min_{0,\varphi} \text{ and } \lim_{a\to 0} T_{n,a,T_1} = T_1$$

holds. That is, the family $\mathcal{J}$ contains the first two ones as limit cases. Moreover, notice that the members of this family change continuously with respect to the parameter a and the following holds true:

$$T_{n,t,T_1} = \min_{0,\varphi} \text{ and } T_{n,0,T_1} = T_1.$$

7 Conclusion

In this paper we characterized the solutions of the annihilation problem (see (1)) under the assumption that T is continuous. The following theorem is established:

Theorem 6 *A continuous t-norm T can be n-annihilated if and only if the annihilated t-norm $T_{(n)}$ is an element of the family $\mathcal{J}$ (see Section 6).*

As a consequence, a new family of t-norms (the nilpotent ordinal sums) was discovered. The residuated implication generated by any member of this family admits the contrapositive symmetry property. Moreover, the nilpotent ordinal sum family (in the extended sense) contains the nilpotent t-norms and the nilpotent minimum as well which have been the only known examples with the above-mentioned property.

A characterization of those continuous Archimedean t-norms which has 0-value exactly when $x \le n(y)$ for a given strong negation n is given. The nilpotent minimum of Fodor is described as limit of nilpotent Archimedean t-norms, which have the above-mentioned property with $n(x) = 1 - x$.

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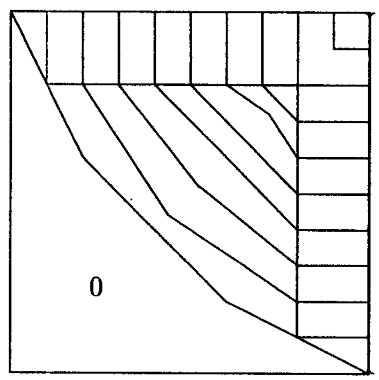

A nilpotent ordinal sum

Figure 1.